STABILITY AND CUT POINTS OF PROBABILISTIC AUTOMATA

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ABSTRACT

STABILITY AND CUT POINTS OF PROBABILISTIC AUTOMATA

by Gerald M. Flachs

The concept of probabilistic automata has recently been the object of much study by automata theorists. The behavior of a probabilistic automaton is essentially characterized by products of matrices selected from a given finite set of stochastic symbol matrices. It is important in many applications that these matrix products be stable with respect to small perturbations of the entries in the symbol matrices. This thesis concerns three different types of stability problems that arise when one considers the effect of these small perturbations upon the behavior of the probabilistic automaton. These are: 1) strict stability, denoted "s-stability"; 2) tape acceptance stability, denoted "a-stability"; 3) zero stability, denoted "0-stability".

Strict stability is concerned with the asymptotic behavior of long products of stochastic matrices whose entries are subjected to small perturbations. Necessary and sufficient conditions are given for an arbitrary probabilistic automaton to be strictly stable. An effective algorithm is given for deciding whether or not an arbitrary automaton is strictly stable.

Tape acceptance stability is concerned with the tape acceptance behavior when the entries in the symbol matrices are subjected to small perturbations. Sufficient conditions are given for a-stability

in terms of s-stability. Also, sufficient conditions are given for a-stability that do not require s-stability. This result is essentially a regional stability result that gives the size of perturbations allowed without causing a-instability.

Zero stability, subject of the major contributions of this thesis, is concerned with the strict stability problem when the perturbations are not allowed to change the zero entry configurations of the symbol matrices. Zero stability results are given in terms of the cyclic structure of probabilistic automata. The fundamental properties of the cyclic structure are developed and refined in order to obtain some 0-stability results. Zero stability results are also given in terms of the algebraic structure of probabilistic automata, which is developed along definite algebraic lines. An important class of probabilistic automata, called "zero-reset" automata, and including group automata, is shown to be 0-stable.

Finally, isolated cut-point problems are discussed using two different approaches. In a set theoretic approach, a set of response intervals is defined which contain the response points. In a topological approach, a pseudo-closure operator is defined that encloses the points which are not isolated cut points. Several tests are given for solving these problems for a large class of automata.

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TABLE OF CONTENTS

I. INTRODUCTION

The concept of a probabilistic automaton has received a great deal of attention due to its evident relationship with the reliability of deterministic automata (Rabin [13], 1963). More recently, the study of neural nets and decision computers has led to an even greater interest in the behavior of probabilistic automata (Kilmer and McCulloch [10], 1964). Essentially, the behavior of a probabilistic automaton is characterized by properties of products of stochastic matrices selected from a finite set of symbol matrices, and subjected to special start and stop conditions.

It is especially important in decision making, that the behavior of the automaton be stable with respect to small fluctuations in the entries of the symbol matrices. That is to say, the behavior of the automaton should not change erratically under small perturbations of the entries in the symbol matrices. Chapter 2 and 3 pertain to this stability problem. Chapter 2 establishes the fundamental properties of stochastic matrices and their products, while reviewing the known stability results. The "strict" stability problem is solved, which allows any entry in the symbol matrices to be perturbed. Chapter 3 is concerned with the restricted zero stability problem, in which the zero configurations in the symbol matrices are not allowed to be perturbed. Zero stability results are given in terms of the cyclic and algebraic structure of probabilistic automata.

Tape acceptance stability and equivalence of deterministic and probabilistic automata depend on the existence of an isolated cut point (Rabin [13], 1963). Chapter 4 focuses on the existence of isolated cut points and on tests which decide whether or not a given cut point is isolated. A bounded algorithm is given which decides these problems for a large class of automata.

1.1. The Probabilistic Automaton Concept

Rabin, in 1963, gave the first neat definition of a probabilistic automaton as a generalization of the usual deterministic automaton. His formulation was essentially as follows. Let $\mathcal{S}_n$ be the set of all $(1 \times n)$ stochastic vectors.

<u>Definition 1.1.1.</u>: A probabilistic automaton is a system $\mathcal{P} = \mathcal{P}(S, \mathcal{M}, \pi_o, O_F)$ defined over a finite alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_{o\Sigma}\}$ where ${}^o\Sigma$ denotes the order of the set Σ and

$S = \{s_1, s_2, \ldots, s_n\}$ is a finite set of states.

π_o is a $(1 \times n)$ stochastic row vector called the initial distribution. Rabin, in 1963, used a single start state instead of an initial distribution. The former has been used in the most recent works of A. Paz, C. Page, and others.

$\mathcal{M} = \{M(\sigma_i) : \sigma_i \in \Sigma\}$ is a finite set of $n \times n$ stochastic matrices $M(\sigma_i)$ that define, for each symbol σ_i in the given set Σ, a mapping from a distribution $\pi \in \mathcal{S}_n$ to a distribution $\pi M(\sigma_i) \in \mathcal{S}_n$. Thus $\mathcal{M}$ is a set of mappings from $\mathcal{S}_n \times \Sigma$ to $\mathcal{S}_n$. Frequently $M(\sigma_i)$

will be called a symbol matrix.

O_F is a $(n \times 1)$ column output vector with ones corresponding to the designated final states $F \subset S$ and zeros elsewhere.

We denote by Σ^* the set of all finite sequences of elements from Σ. We call these elements of Σ^* tapes, and we denote Σ_N and Σ^N to be the set of tapes in Σ^* whose lengths are N and no larger than N respectively. The length q of the tape $x = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_q} \in \Sigma^*$ is denoted by $\lg(x)$. We also write $x_k^p = \sigma_{i_k} \sigma_{i_{k+1}} \ldots \sigma_{i_p}$ to denote the subtape of x that consists of the k^{th} symbol through the p^{th} symbol.

The function $\mathcal{M}: \mathcal{S}_n \times \Sigma \rightarrow \mathcal{S}_n$, defined for symbols $\sigma_i \in \Sigma$, admits a natural extension to $\mathcal{M}: \mathcal{S}_n \times \Sigma^* \rightarrow \mathcal{S}_n$, defined for tapes $x = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_q}$ by the matrix product $\pi M(x) = \pi M(\sigma_{i_1}) M(\sigma_{i_2}) \ldots M(\sigma_{i_q})$. Here the null tape Λ $(\lg(\Lambda) = 0)$ is represented by the identity matrix $M(\Lambda) = I$, and the mapping $\pi I = \pi$.

The behavior of probabilistic automata can be viewed as a stimulus-response relation for a mathematical machine. This point of view is particularly important in the application of probabilistic automata to animal and antefactual decision behavior. The response of a probabilistic automaton $\mathcal{P}$ to a sequence of stimuli, $x = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_k}$, is defined by

$$rp(x) = \pi_o M(\sigma_{i_1}) M(\sigma_{i_2}) \ldots M(\sigma_{i_k}) O_F .$$

The response $rp(x)$ of ρ to a tape x is the probability of entering into a final state upon the application of tape x to ρ when ρ is started with the initial distribution π_o. We shall often refer to $rp(x)$ as a "response point". Thus we see that the response of a probabilistic automaton is characterized by products of stochastic matrices selected from the symbol matrices together with special start and stop conditions, π_o and O_F respectively.

The tape acceptance behavior of a probabilistic automaton is defined in terms of a cut point $0 \leq \lambda \leq 1$. For a given cut point λ, the system (ρ, λ) is said to <u>accept</u> the set of tapes $T(\rho, \lambda)$,

$$T(\rho, \lambda) = \{x : x \in \Sigma^*, \ rp(x) > \lambda\}$$

and <u>reject</u> the rest, all with respect to the cut point λ. A cut point λ is said to be isolated if, for every $x \in \Sigma^*$ and some $\gamma > 0$, either $rp(x) > \lambda + \gamma$ or $rp(x) < \lambda - \gamma$.

<u>Definition 1.1.2.</u>: A cut point $0 \leq \lambda \leq 1$ is called isolated iff for some fixed $\gamma > 0$, $|rp(x) - \lambda| > \gamma$ for all $x \in \Sigma^*$. We shall often refer to a γ-isolated cut point to signify that we have a particular fixed γ in mind.

Rabin showed that probabilistic automata with isolated cut points accept only those sets of tapes that are definable by finite state deterministic automata. Thus probabilistic automata with isolated cut points have the same tape discrimination power as finite state deterministic automata. A probabilistic automaton, however, may have vastly fewer states than any corresponding deterministic automaton accepting the same set of tapes.

II. GENERAL STABILITY PROBLEM

This chapter launches our attack on the stability problem and offers a review of the known stability results. We discuss three different types of stability that are of interest in probabilistic automata theory. The first type, defined without reference to a cut point, is strictly concerned with the asymptotic behavior of long products of stochastic matrices. The second type, defined in terms of a cut point, is concerned with the tape acceptance behavior of probabilistic automata. The third type, called zero stability and discussed in chapter three, concerns the asymptotic behavior of long products of stochastic matrices whose entries are subjected to perturbations which do not alter any matrix zero configuration. We shall define these stability concepts precisely below.

2.1. Stability Concepts

The stability concepts introduced here pertain to the Rabin probabilistic automaton $\mathcal{P}(S, \mathcal{M}, \pi_o, O_F)$ defined in Section 1.1. Stability problems arise when one considers the behavior of a probabilistic automaton under small perturbations of the entries in its symbol matrices $M(\sigma_i)$, $\sigma_i \in \Sigma$. Only those perturbations which leave the perturbed symbol matrices stochastic are allowed. We shall denote the perturbation of $\mathcal{P}(S, \mathcal{M}, \pi_o, O_F)$ by $\mathcal{P}'(S, \mathcal{M}', \pi_o, O_F)$. That is, $\mathcal{P}'$ is a system $\mathcal{P}'(S, \mathcal{M}', \pi_o, O_F)$ in which the entries of each symbol matrix $M'(\sigma_i)$, $\sigma_i \in \Sigma$, are formed by perturbing the entries of $M(\sigma_i)$, $\sigma_i \in \Sigma$, by arbitrary small quantities that leave the row sums one.

Let $|B|$ denote the absolute value of the maximum entry in B.

<u>Definition 2.1.1.</u>: An automaton $\mathcal{P}(S, \mathcal{M}, \pi_o, O_F)$ is strictly stable (denoted <u>s-stable</u>) iff given any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that the inequalities

$$|M(\sigma_i) - M'(\sigma_i)| < \delta \qquad \forall \ \sigma_i \in \Sigma$$

imply

$$|M(x) - M'(x)| < \epsilon \qquad \forall \ x \in \Sigma^* .$$

<u>Definition 2.1.2.</u>: An automaton $\mathcal{P}(S, \mathcal{M}, \pi_o, O_F)$ with cut point λ is tape-acceptance stable (denoted <u>a-stable</u>) iff there exists a $\delta > 0$ such that the inequalities

$$|M(\sigma_i) - M'(\sigma_i)| < \delta \qquad \forall \ \sigma_i \in \Sigma$$

imply

$$T(\mathcal{P}, \lambda) = T(\mathcal{P}', \lambda) .$$

In other words, a probabilistic automaton with cut point λ is a-stable if its accepted tape set is not changed by sufficiently small perturbations of its symbol matrices.

<u>Definition 2.1.3.</u>: An automaton $\mathcal{P}(S, \mathcal{M}, \pi_o, O_F)$ is zero stable (denoted o-stable) iff given any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that the two conditions

1) $\quad |M(\sigma_i) - M'(\sigma_i)| < \delta \ , \quad \forall \ \sigma_i \in \Sigma \ ,$

2) no perturbation is allowed to change the zero entry configuration in any $M(\sigma_i)$, $\sigma_i \in \Sigma$,

imply the inequalities

$$|M(x) - M'(x)| < \epsilon \qquad \forall \ x \in \Sigma^* .$$

The following simple example illustrates these concepts by showing the difference between o-stability and s-stability.

Example 2.1.1.: We consider an automaton that is, 1) o-stable, 2) not s-stable, and 3) for $0 < \lambda < 1/2$ not a-stable. Define $\mathcal{P}(S, \mathcal{M}, s_1, s_2)$ with cut point $0 < \lambda < 1/2$ over the single symbol alphabet $\Sigma = \{0\}$. Let the state set be $S = \{s_1, s_2\}$, and let

$$M(0) = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} .$$

1) First note that $\mathcal{P}$ is o-stable, since $\mathcal{P}' = \mathcal{P}$ when no zero entries are altered.

2) Next, we shall prove that $\mathcal{P}$ is s-unstable. We show that a perturbation for which $0 < \delta < 1$ will introduce a significant change in the asymptotic behavior of $M^k(0)$. Consider the perturbed system $\mathcal{P}'$ with transition matrix

$$M'(0) = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} 1-\delta & \delta \\ \delta & 1-\delta \end{bmatrix} \end{array}$$

where $0 < \delta < 1$. For the tape $x = 0^k \in \Sigma^*$, the perturbed matrix product is

$$M'(x) = M'(0^k) = \begin{bmatrix} 1-\delta & \delta \\ \delta & 1-\delta \end{bmatrix}^k .$$

The matrix $M'(0)$ can be written in terms of its constituent matrices as

$$M'(0) = U_1 + (1 - 2\delta)U_2$$

where

$$U_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} .$$

Since $U_1^2 = U_1$, $U_2^2 = U_2$ and $U_1U_2 = 0$, we see by induction that

$$M'(0^k) = U_1 + (1 - 2\delta)^k U_2 .$$

Now $(1 - 2\delta)^k \to 0$ as k increases; thus, given any positive $\epsilon < \frac{1}{2}$ there exists a finite integer $K(\delta, \epsilon)$ (any integer $K \geq \epsilon/\delta(1-2\epsilon)$ will do) such that

$$\begin{aligned} |M(0^K) - M'(0^K)| &= |U_2| \cdot [1 - (1 - 2\delta)^K] \\ &= \frac{1}{2}[1 - (1 - 2\delta)^K] > \epsilon \end{aligned} \qquad (2.1.1)$$

Hence it follows that $\mathcal{P}$ is s-unstable.

3) Finally, for any cut point λ, $0 < \lambda < \frac{1}{2}$ we have $T(\mathcal{P}, \lambda) = \phi$. The set $T(\mathcal{P}', \lambda)$ is not null, however, since

given any $\delta > 0$ there exists by equation (2.1.1) a finite integer $K(\delta, \lambda)$ such that $r\, p_{\rho'}(0^K) > \lambda$ and $r\, r_{\rho}(0^K) = 0 < \lambda$. Hence ρ is a-unstable.

2.2. Stochastic Matrices and Their Products

In this section, we summarize some fundamental results concerning stochastic matrices and their products.

Definition 2.2.1.: The $(n \times 1)$ column vector with all 1 entries is denoted I_s and called the "summing vector."

Definition 2.2.2.: A square matrix A is a stochastic matrix iff it has nonnegative entries and unit row sums.

- The nonnegative (positive) entry condition is written $A \geq 0$ $(A > 0)$.
- The unit row sum condition is written $A\, I_s = I_s$.

Lemma 2.2.1.: If A and B are two $(n \times n)$ stochastic matrices, then the product $C = AB$ is again a stochastic matrix.

Proof: The nonnegative conditions $A \geq 0$ and $B \geq 0$ imply

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k}\, b_{k,j} \geq 0\,, \quad C \geq 0\,.$$

The unit row sum conditions $AI_s = I_s$ and $BI_s = I_s$ imply

$$C \cdot I_s = A\,(B\, I_s) = A\, I_s = I_s\,.$$

Lemma 2.2.2.: The eigenvalues λ_i of a $(n \times n)$ stochastic matrix A satisfy $|\lambda_i| \leq 1$.

Proof: If λ is any eigenvalue of A with corresponding eigenvector x then $Ax = \lambda x$. Let x_i be the component of the largest modulus of x. Now consider the modulus of the i^{th} row of $\lambda x = Ax$,

$$|\lambda x_i| = |\sum_{j=1}^{n} a_{i,j} x_j| \leq \sum_{j=1}^{n} a_{i,j} |x_i| = |x_i| ,$$

which implies $|\lambda| \leq 1$.

Lemma 2.2.3.: Any $(n \times n)$ stochastic matrix A has at least one unit eigenvalue with corresponding eigenvector I_s.

This result is an immediate consequence of the unit row sum condition, $A I_s = I_s$.

Definition 2.2.3.: An $(n \times n)$ stochastic matrix A is scrambling iff $A A^T > 0$. Equivalently, a stochastic matrix A is scrambling iff every pair of rows (i, r) of A has a corresponding column j such that $a_{i,j} > 0$ and $a_{r,j} > 0$.

- Let $||C||$ denote the maximum difference $|c_{i,j} - c_{r,j}|$ for all i, r, j.

Theorem 2.2.1.: (Equivalent to a theorem of Paz [12]). If A and B are $(n \times n)$ stochastic scrambling matrices, then $\| A \| \leq (1 - a_{min}) \| B \|$ where a_{min} is the minimal nonzero entry in A.

Proof: For an arbitrary fixed j, let i and r be the integers that represent the particular two rows of A that generate the j^{th} column norm $\| (A\ B)_{\cdot,j} \|$. Then we have

$$\|(A\, B)_{\cdot, j}\| = \left| \sum_{k=1}^{n} (a_{i,k}\, b_{k,j} - a_{r,k}\, b_{k,j}) \right|$$

$$= \left| \sum_{k=1}^{n} (a_{i,k} - a_{r,k})\, b_{k,j} \right| \quad .$$

Let K and $\overline{K}$ be the sets of indices k such that $a_{i,k} \geq a_{r,k}$ and $a_{i,k} < a_{r,k}$ respectively. Define

$$\Sigma^{+} = \sum_{k \epsilon K} (a_{i,k} - a_{r,k}) \geq 0 \quad \text{and} \quad \Sigma^{-} = - \sum_{k \epsilon \overline{K}} (a_{i,k} - a_{r,k}) \geq 0 \quad .$$

The unit row sum condition implies that

$$\Sigma^{+} = \Sigma^{-} \quad .$$

The quantity Σ^{+} satisfies the condition

$$\Sigma^{+} = \sum_{k \epsilon K} (a_{i,k} - a_{r,k}) = 1 - \sum_{k \epsilon K} a_{r,k} - \sum_{k \epsilon \overline{K}} a_{i,k} \leq 1 - a_{min}$$

since the scrambling condition insures that at least one term within the sums must be positive. Finally, we conclude that

$$\|(A \cdot B)_{\cdot, j}\| = \left| \sum_{k \epsilon K} (a_{i,k} - a_{r,k})\, b_{k,j} + \sum_{k \epsilon \overline{K}} (a_{i,k} - a_{r,k})\, b_{k,j} \right|$$

$$\leq \left| \sum_{k \epsilon K} (a_{i,k} - a_{r,k})\, b^{j}_{max} - \sum_{k \epsilon \overline{K}} (a_{r,k} - a_{i,k})\, b^{j}_{min} \right|$$

$$= \left| \Sigma^{+}\, b^{j}_{max} - \Sigma^{-}\, b^{j}_{min} \right|$$

$$\leq \Sigma^{+} \cdot \|B\| \leq (1 - a_{min}) \cdot \|B\| \quad .$$

This completes the proof.

The stronger inequality $\|A \cdot B\| \leq \|A\| \cdot \|B\|$ might be conjectured, but is not satisfied by the two matrices

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix} .$$

Lemma 2.2.4.: If A is an $(n \times n)$ stochastic matrix and B is any $(n \times n)$ matrix, then $|AB - B| \leq \|B\|$.

Proof: We let β be an $(n \times 1)$ column vector defined as $\beta^T = (\beta_1, \beta_2, \ldots, \beta_n)$ and show that $|A\beta - \beta| \leq \|\beta\|$. Let i denote a row which yields $|A\beta - \beta|$, the largest absolute value among the entries of $A\beta - \beta$. Then we see that

$$|A\beta - \beta| = \left|\sum_{k=1}^{n} a_{i,k}\beta_k - \beta_i\right| = \left|\sum_{k=1}^{n} a_{i,k}\beta_k - \beta_i \sum_{k=1}^{n} a_{i,k}\right|$$

$$= \left|\sum_{k=1}^{n} (\beta_k - \beta_i) a_{i,k}\right| \leq \sum_{k=1}^{n} |\beta_k - \beta_i| \, a_{i,k}$$

$$\leq \|\beta\| \cdot \sum_{k=1}^{n} a_{i,k} = \|\beta\| .$$

Since the column vector β can be chosen at will, we conclude that $|AB - B| \leq \|B\|$.

Lemma 2.2.5.: A $(n \times n)$ scrambling stochastic matrix A has exactly one eigenvalue with unit modulus: $\lambda_1 = 1$.

Proof: Since A is scrambling, we know by Theorem 2.2.1 that $\|A^m\| \leq (1 - a_{min})^m$ where a_{min} is the smallest nonzero entry in A. Thus $\lim_{m\to\infty} \|A^m\| = 0$ and hence $U_1 = \lim_{m\to\infty} A^m$ is a stochastic idempotent matrix with identical rows. Consequently, the only nonzero eigenvalue of U_1 is $\lambda_1 = 1$. The eigenvalues of A^m are $\lambda_i^m (i = 1, 2, \ldots, n)$ where λ_i are the eigenvalues of A. Thus, A has precisely one eigenvalue with unit modulus.

This result can be strengthened by introducing a more liberal scrambling condition.

Wolfowitz [16] proved that if A is scrambling,

- so are AB and BA, for stochastic matrices A, B.

<u>Definition 2.2.4.</u>: A $(n \times n)$ stochastic matrix is said to be eventually scrambling (denoted e-scrambling) iff there exists an integer e such that A^e is a scrambling matrix.

<u>Theorem 2.2.2.</u>: A $(n \times n)$ stochastic matrix A is e-scrambling iff A has precisely one eigenvalue, $\lambda_1 = 1$, with unit modulus.

<u>Proof:</u> a) If A is e-scrambling, then by Theorem 2.2.1, it follows that $\lim_{m \to \infty} \|A^m\| = 0$. Arguments similar to those used in Lemma 2.2.5 prove that A has precisely one unit modulus eigenvalue, $\lambda_1 = 1$.

b) If A has only one unit modulus eigenvalue, $\lambda_1 = 1$, then $|\lambda_i| < 1$ for $i = 2, 3, \ldots, n$. This implies that the stochastic idempotent $U_1 = \lim_{m \to \infty} A^m$ has only one unit eigenvalue and $(n - 1)$ zero eigenvalues. Thus U_1 has rank 1. We observe that any stochastic matrix with two non-identical rows has rank at least two. Consequently, $U_1 = I_s \cdot u^T$ has identical rows u^T and $\lim_{m \to \infty} \|A^m\| = 0$. This implies that A is e-scrambling, since there exists a bound B (namely n) such that if A^m is scrambling for any $m > B$, then

A^B is also scrambling.

<u>Definition 2.2.5.</u>: A square matrix A, labeled by states, is said to be <u>reducible</u> iff there exists a relabeling of the states so that the rearranged matrix A_r has the form

$$A_r = \begin{bmatrix} P_r & 0 \\ Q_r & T_r \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} T_r & Q_r \\ 0 & P_r \end{bmatrix}$$

where P_r is a square matrix. Otherwise A is called irreducible.

<u>Definition 2.2.6.</u>: A square matrix A is said to be <u>partially v-decomposable</u> iff there exists a relabeling of the states so that the rearranged matrix A_r has the block form

$$A_r = \begin{array}{c} \\ S^{(o)} \\ S^{(1)} \\ S^{(2)} \\ \vdots \\ S^{(v)} \end{array} \begin{array}{c} \begin{matrix} S^{(o)} & S^{(1)} & S^{(2)} & \cdots & S^{(v)} \end{matrix} \\ \begin{bmatrix} T_r & Q^{(1)} & Q^{(2)} & \cdots & Q^{(v)} \\ & P^{(1)} & & & \\ & & P^{(2)} & & O \\ & O & & \ddots & \\ & & & & P^{(v)} \end{bmatrix} \end{array}$$

where $P^{(i)}$ $(i = 1, 2, \ldots, v)$ are square matrices. If the set $S^{(o)}$ is null, then the matrix A is said to be <u>v-decomposable.</u>

<u>Theorem 2.2.3.</u>: (Perron-Frobenius) If A is any nonnegative irreducible square matrix, then:

1) there exists a real positive eigenvalue λ_1 of A such that, if λ is any other eigenvalue of A then

$|\lambda| \leq \lambda_1$;

2) minimal row sum $\leq \lambda_1 \leq$ maximal row sum, where neither equality holds unless the row sums of A are equal;

3) there exists a real positive eigenvector $X_1 > 0$ such that $A X_1 = \lambda_1 X_1$;

4) λ_1 increases when any entry in A increases; and

5) λ_1 is a simple root.

This result was also proved by G. Debrew and I. N. Herstein [6] using Brouwer's fixed point theorem. However, their proof does not give a constructive method for determining λ_1 .

Proof: We shall first prove parts 1), 2), 3), and 4) by showing that there exists a sequence of diagonal similarity transformations whose limit transforms A into a matrix with equal positive row sums. The approach taken gives a constructure procedure for determining λ_1. The essential features of the approach can be clarified by means of a simple example.

Example: Consider the nonnegative irreducible matrix A:

$$A = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 0 & 5 \\ 3 & 4 & 4 \end{bmatrix} , \quad A I_s = \begin{bmatrix} 7 \\ 5 \\ 11 \end{bmatrix} .$$

We shall apply successive diagonal similarity transformations that modify the rows with maximal or minimal row sum, so as to increase the minimal row sum and/or decrease the maximal row sum. First, since the third row has the maximum row sum 11, we transform A by a diagonal matrix $D_1 = \text{diag}\{1, 1, d_1\}$ and obtain

$$A_1 = D_1^{-1} A D_1 = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 0 & 5d_1 \\ 3/d_1 & 4/d_1 & 4 \end{bmatrix}$$

We now choose $d_1 > 0$ to satisfy the inequalities

$$5 < 5d_1 < 11 \quad , \text{ or } \quad 1 < d_1 < 11/5$$
$$5 < 4 + 7/d_1 < 11 \; , \text{ or } \quad 1 < d_1 < 7$$

so as to obtain a smaller maximum row sum. We choose $d_1 = 2$, whence

$$A_1 = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 0 & 10 \\ 3/2 & 2 & 4 \end{bmatrix} \quad \begin{matrix} \text{(Row Sum)} \\ (7) \\ (10) \\ (7.5) \end{matrix}$$

Next, we transform A_1 by a diagonal matrix $D_2 = \text{diag}\{1, d_2, 1\}$ and obtain

$$A_2 = D_2^{-1} A_1 D_2 = \begin{bmatrix} 3 & 4d_2 & 0 \\ 0 & 0 & 10/d_2 \\ 3/2 & 2d_2 & 4 \end{bmatrix} .$$

We now choose $d_2 > 0$ to satisfy the inequalities

$$7 < 3 + 4d_2 < 10 \quad \text{or} \quad 1 < d_2 < 7/4$$
$$7 < 10/d_2 < 10 \quad \text{or} \quad 1 < d_2 < 10/4$$
$$7 < 5.5 + 2d_2 < 10 \quad \text{or} \quad 3/4 < d_2 < 9/4$$

so as to obtain a smaller maximum row sum. With the judicious choice $d_2 = 5/4$, we get a similar matrix A_2 with equal row sums.

$$A_2 = \begin{bmatrix} 3 & 5 & 0 \\ 0 & 0 & 8 \\ 3/2 & 5/2 & 4 \end{bmatrix} \begin{matrix} (8) \\ (8) \\ (8) \end{matrix} .$$

Consequently, A has a positive real eigenvalue $\lambda_1 = 8$. Clearly, the matrix $(1/8)A_2$ is stochastic. If λ is any other eigenvalue of A_2 with eigenvector Y then $A_2 Y = \lambda Y$. Dividing by $\lambda_1 = 8$ we get $(1/8)A_2 Y = (\lambda/8) Y$. By Lemma 2. 2. 2, the modulus of any eigenvalue of a stochastic matrix cannot exceed one. Consequently, $|\lambda| \leq 8$ for any eigenvalue λ of A.

We now return to the proof of the theorem at hand. First, we establish a sequence of diagonal similarity transformations, $D_i^{-1} A_i D_i = A_{i+1}$ $(i = 0, 1, \ldots)$ where $A_0 = A$, so that $\lim_{i\to\infty} \|A_i I_s\| \to 0$. We shall alternately operate on the maximal and minimal row sums. Thus for i even (i odd) we will operate on the maximal (minimal) row sums. Denote the rows with maximal, intermediate, and minimal row sums by K^+, K^o and K^- respectively. Define the diagonal matrices $D_i = \text{diag}\{d_j^{(i)}\}$

<u>for i even as</u>

$$d_j^{(i)} = d_i > 1 \quad \text{if} \quad j \in K^+$$

$$d_j^{(i)} = 1 \qquad \text{otherwise} \ , \ \text{and}$$

for i odd as

$$0 < d_j^{(i)} = d_i < 1 \quad \text{if } j \in K^-$$
$$d_j^{(i)} = 1 \qquad \text{otherwise.}$$

Now to prove that at each step $d_i > 0$ can be chosen so that $\lim_{i\to\infty} \|A_i I_s\| = 0$, we note that at each step the matrix $D_i^{-1} A_i D_i = A_{i+1}$ can be relabeled so that the relabeled matrix A_{i+1} has the form

for i even:

$$A_{i+1}^r = \begin{array}{c} \\ K^+ \\ K^0 \\ K^- \end{array} \begin{array}{c} \begin{array}{ccc} K^+ & K^0 & K^- \end{array} \\ \begin{bmatrix} A_{11} & (1/d_i)A_{12} & (1/d_i)A_{13} \\ d_i A_{21} & A_{22} & A_{23} \\ d_i A_{31} & A_{32} & A_{33} \end{bmatrix} \end{array} \quad \text{and,}$$

for i odd:

$$A_{i+1}^r = \begin{array}{c} \\ K^+ \\ K^0 \\ K^- \end{array} \begin{array}{c} \begin{array}{ccc} K^+ & K^0 & K^- \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & d_i A_{13} \\ A_{21} & A_{22} & d_i A_{23} \\ (1/d_i)A_{31} & (1/d_i)A_{32} & A_{33} \end{bmatrix} \end{array} .$$

The irreducible condition implies for even i that $d_i > 0$ can be chosen so that; 1) at least one of the maximal row sums will be decreased, 2) at least one of the row sums in $K^0 \cup K^-$ will be increased. Similarily, for i odd we can choose $d_1 > 0$ so that; 1) at least one of the minimal row sums will be increased, 2) at least one or the row sums in $K^+ \cup K^0$ will be decreased. Consequently, as the

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process is repeated, all the row sums will tend toward a common positive row sum λ_1. The quantity λ_1 lies between the minimal and maximal row sums, but is equal to either only when both are equal.

Consider a sequence of diagonal similarity transformations such that

$$\lim_{n\to\infty} \|(D_o D_1 D_2 \ldots D_n)^{-1} A (D_o D_1 D_2 \ldots D_n) I_s\|$$
$$= \|D^{-1} A D I_s\| = 0 .$$

This simply states that the row sums of $Q = D^{-1} A D$ have equal values λ_1. Thus, this matrix Q similar to A satisfies $Q I_s = \lambda_1 I_s$, so λ_1 is an eigenvalue of A. Now $(1/\lambda_1) Q$ is a stochastic matrix. If λ is any other eigenvalue of Q with eigenvector Y then $Q Y = \lambda Y$. Dividing by λ_1 we get $(1/\lambda_1) Q Y = (\lambda/\lambda_1) Y$. By Lemma 2. 2. 2, the modulus of any eigenvalue of a stochastic matrix can not exceed one. Consequently, $|\lambda| \leq \lambda_1$ for any eigenvalue λ of A.

To prove that there exists a positive eigenvector corresponding to λ_1, we note that $D^{-1} A D I_s = \lambda_1 I_s$ where $D > 0$. Consequently $X_1 = D I_s > 0$ is a positive eigenvector of A corresponding to λ_1.

To prove that λ_1 increases when any element in A is increased, one needs only to observe that a row sum of $D^{-1} A D$ increases when any entry in A increases. This completes the proof of parts 1), 2), 3), and 4).

To prove that λ_1 is a simple root of $D(\lambda) = \det [I\lambda - A]$ note by [7] that $D'(\lambda) = tr(B(\lambda))$ where $B(\lambda)$ denotes the adjoint of $(I\lambda - A)$.

Let C_i be the $(n - 1) \times (n - 1)$ submatrix obtained from A by deleting the i^{th} row and column. Then we know from the adjoint definition that

$$D'(\lambda) = \mathrm{tr}\,(B(\lambda)) = \sum_{i=1}^{n} \det\,(\lambda\, I - C_i)$$

By 4) it is clear that $\det\,(\lambda I - C_i) > 0$ for $\lambda \geq \lambda_1$. Consequently, $D'(\lambda) > 0$ for $\lambda \geq \lambda_1$ which proves λ_1 is a simple root of $\det\,(I\lambda - A)$. This completes the proof of the theorem.

The next theorem shows the relationship between the number of unit eigenvalues and the structure of a stochastic matrix.

Theorem 2.2.4.: A $(n \times n)$ stochastic matrix A has v unit eigenvalues iff A is partially v-decomposable.

Proof: a) (Sufficiency) We prove that if A has v unit eigenvalues then A is partially v-decomposable. First, we perform a suitable relabeling of the states such that A has the block upper triangular form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ & A_{22} & \cdots & A_{2n} \\ & & \ddots & \vdots \\ & & & A_{nn} \end{bmatrix}$$

where the square matrices A_{ii} are irreducible. Clearly any eigenvalue of A must be an eigenvalue of some A_{ii} and vice-versa. Conclusion 4 of Theorem 2.2.3 proves that, if A_{ii} has a unit modulus eigenvalue, then $A_{ij} = 0$ for $j = i + 1, \ldots, n$, since the eigenvalues of any stochastic matrices have modulus no larger than one. Consequently, A_{ii} is a stochastic irreducible matrix with only one unit eigenvalue. Thus, if A has v-unit eigenvalues, then A is partially v-decomposable.

b) The proof of necessity follows trivially from Lemma 2. 2. 3.

This result can be extended to nonnegative square matrices with row sums no larger than one. Such matrices we shall call substochastic.

Theorem 2. 2. 5.: A (n x n) substochastic matrix A has v unit eigenvalues iff A is partially v-decomposable.

The proof follows immediately from Theorem 2. 2. 4, by observing that a substochastic matrix A can be imbedded into a stochastic matrix $\hat{A}$ by adjoining one additional state as shown below:

$$\hat{A} = \left[\begin{array}{ccc|c} & & & d_1 \\ & & & d_2 \\ & A & & \vdots \\ & & & d_n \\ \hline 0 & \dots & 0 & 1 \end{array}\right]$$

where the d's are chosen to produce unit row sums.

2. 3. Strict Stability

The results obtained in this section are sufficient to solve the strict stability problem for an arbitrary probabilistic automaton.

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We shall prove that the quasi-definite condition is both necessary and sufficient for strict stability. An efficient algorithm will be given in Chapter 3 to decide whether or not a given probabilistic automaton is quasi-definite. First the quasi-definite condition is defined as a non-trivial generalization of Rabin's actual automaton.

Definition 2. 3. 1.: A probabilistic automaton $\mathcal{P}(S, \mathcal{M}, \pi_o, 0_F)$ is called actual iff the symbol matrices $M(\sigma_i)$, $\sigma_i \in \Sigma$ contain no zero entries.

Rabin proved that all actual probabilistic automata have the property that given any $\epsilon > 0$ there exists an integer $N(\epsilon)$ such that the inequality $lg(x) \geq N$, $x \in \Sigma^*$, implies $\|M(x)\| < \epsilon$. This result follows immediately from Theorem 2. 2. 1, since the actual condition clearly implies that all the symbol matrices are scrambling. A class of automata that includes these actual automata will be called quasi-definite automata, following A. Paz.

Definition 2. 3. 2.: A probabilistic automaton $\mathcal{P}(S, \mathcal{M}, \pi_o, 0_F)$ is called quasi-definite iff given $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that the inequality $lg(x) \geq N$, $x \in \Sigma^*$, implies $\|M(x)\| < \epsilon$.

Theorem 2. 3. 1.: A probabilistic automaton $\mathcal{P}(S, \mathcal{M}, \pi_o, 0_F)$ is s-stable iff it is quasi-definite.

Proof: a) (Sufficiency) We shall prove, for any tape $x \in \Sigma^*$, that $|M(x) - M'(x)| < \epsilon$ if the perturbations of the symbol matrices are made sufficiently small. If $\mathcal{P}$ is quasi-definite, there exists an integer $N(\epsilon)$ such that $lg(x) \geq N$, $x \in \Sigma^*$ implies $\|M(x)\| < \epsilon/4$.

We now consider any tape $x \in \Sigma^*$. If $lg(x) \leq N$, then we clearly can choose our perturbations sufficiently small that $lg(x) \leq N$ implies $|M(x) - M'(x)| < \epsilon/8 < \epsilon$. If $lg(x) > N$, then we partition x so that

$$x = y\, z$$

where $lg(z) = N$. Now consider

$$|M(x) - M'(x)| = |M(y)M(z) - M'(y)M'(z)|$$

from which we get

$$|M(x) - M'(x)| = |(M(y)\, M(z) - M(z)) - (M'(y)M'(z) - M'(z)) + M(z) - M'(z))|$$

by adding and subtracting terms. The triangle inequality yields

$$|M(x) - M'(x)| \leq |M(y)M(z) - M(z)| + |M'(y)M'(z) - M'(z)| + |M(z) - M'(z)| \; .$$

Applying Lemma 2.2.4, we get

$$|M(x) - M'(x)| \leq \|M(z)\| + \|M'(z)\| + |M(z) - M'(z)| \; .$$

We note that

$$\|M'(z)\| \leq \|M(z)\| + 2\,|M(z) - M'(z)|$$

by observing the entry in $M(z)$ which produces $\|M(z)\|$. Hence, we conclude that there exist sufficiently small perturbations to imply

$$|M(x) - M'(x)| \leq \epsilon/4 + \epsilon/2 + \epsilon/8 = 7/8\, \epsilon < \epsilon$$

for all $x \in \Sigma^*$. This completes the sufficiency portion of the proof.

b) (Necessity) We now show that if ρ is not quasi-definite, then ρ is s-unstable. If ρ is not quasi-definite, there exists a fixed $\gamma > 0$ and an unbounded sequence of tapes $\{x_i\}$ where $lg(x_i) \geq i$ such that for all i, $\|M(x_i)\| > \gamma$. Clearly we can perturb ρ with arbitrarily small nonzero quantities so that the perturbed system ρ' is quasi-definite. This implies that for $\epsilon > 0$ there exists an integer $N(\epsilon)$ such that the condition $lg(x) \geq N$, $x \in \Sigma^*$ implies $\|M'(x)\| < \epsilon$. Hence $\lim_{i\to\infty} \|M'(x_i)\| = 0$, and consequently $\lim_{i\to\infty} |M(x_i) - M'(x_i)| \geq \gamma/2$, so ρ is s-unstable.

Paz introduced a necessary and sufficient condition for quasi-definite automata by his H_4-condition, decidable by a bounded experiment.

<u>Definition 2.3.3.</u>: A probabilistic automaton is said to satisfy the <u>H_4-condition</u> iff there exists an integer k such that $lg(x) \geq k$, $x \in \Sigma^*$, implies that $M(x)$ is scrambling.

<u>Theorem 2.3.2.</u>: A probabilistic automaton is quasi-definite iff it satisfies the H_4-condition.

<u>Proof</u>: a) (Necessity) If a probabilistic automaton is quasi-definite then for $\epsilon > 0$ there exists an integer $N(\epsilon)$ such that $\|M(x)\| < \epsilon$ for all $x \in \Sigma^*$ with $lg(x) \geq N$. This simply means that the rows of $M(x)$ become nearly identical whenever $lg(x) \geq N$, which in turn clearly implies the scrambling condition.

b) (Sufficiency) If a probabilistic automaton satisfies the H_4-condition, then there exists an integer N such that $lg(x) \geq N$ implies $M(x)$ is scrambling. Define $\gamma > 0$ to be the minimal nonzero entry in $\{M(y) \mid y \in \Sigma^N\}$. Now select the minimal integer B such that $(1-\gamma)^B < \epsilon$. Let x be any tape in Σ^* with $lg(x) \geq BN$. Partition x into "prime" scrambling tapes z_i such that the matrices $M(z_i)$ are scrambling and no z_i has a scrambling subtape;

$$x = z_1 z_2 z_3 \ldots z_n , \quad (n \geq B) .$$

The H_4-condition insures that $lg(z_i) \leq N$. Now apply Theorem 2.2.1 on the subtapes z_i to get

$$\|M(x)\| \leq (1-\gamma)^n \leq (1-\gamma)^B < \epsilon .$$

Paz proved [12] that the H_4-condition can be decided with a bounded experiment. This result will be obtained easily from the cyclic structure developed in the next chapter. Also, at that time, an efficient algorithm will be given to decide the quasi-definite condition.

2.4. Acceptance Stability

We shall give here some sufficient conditions for tape acceptance stability expressed in terms of an isolated cut point. A cut point λ is γ-isolated if the response of tape x satisfies $|rp(x) - \lambda| \geq \gamma > 0$ for all $x \in \Sigma^*$. The a-stability differs from s-stability by the fact that a-stability will tolerate the instability of the response points as long as they do not cross the cut point, but will not tolerate the crossing of the cut point by a response point.

Theorem 2.4.1.: If an automaton $\mathcal{P}$ is s-stable and λ is an isolated cut point, then the system $(\mathcal{P}, \lambda)$ is a-stable.

Proof: Let λ be a γ-isolated cut point, i.e., $|rp(x) - \lambda| \geq \gamma$ for all $x \in \Sigma^*$. By Theorem 2.3.1, for $\epsilon > 0$ we can choose the perturbations sufficiently small so that $|M(x) - M'(x)| < \epsilon = \gamma/n$ where n is the number of states in $\mathcal{P}$. Let x be any tape in Σ^* and consider the change in the acceptance resulting from these perturbations:

$$|rp(x) - rp'(x)| = |\pi_o M(x) 0_F - \pi_o M'(x) 0_F|$$

$$= |\pi_o (M(x) - M'(x)) 0_F| \leq n\epsilon = \gamma .$$

This proves that $x \in T(\mathcal{P}, \lambda) \leftrightarrow x \in T(\mathcal{P}', \lambda)$ since no response point can cross the γ-isolated cut point λ.

Our next result is a regional stability theorem that does not require the automaton to be strictly stable in order to be a-stable. That is to say, we will tolerate instability of response points as long as they do not cross the cut point. Let us consider a probabilistic automaton $\mathcal{P}(S, \mathcal{M}, \pi_o, O_F)$ defined over the alphabet $\Sigma = \{1, 2, \ldots, {}^o\Sigma\}$ and $S = \{1, 2, \ldots, {}^oS\}$ where $F \subseteq S$ represents the set of designated final states, and 0_F the column vector with entries 1 for the final states and 0 elsewhere. Define the response intervals, R_i, as

$$R_i = [\sum_{j\in F} \Delta_j^i, \sum_{j\in F} \nabla_j^i] \cap [0, 1] \qquad (2.4.1)$$

for each $i \in \Sigma$ where

$$\Delta_j^i = \min_k \{M(i)_{k,j}\}$$

and

$$\nabla_j^i = \max_k \{M(i)_{k,j}\} .$$

<u>Theorem 2.4.2.</u>: A probabilistic automaton $\mathcal{P}(S, \mathcal{M}, \pi_o, O_F)$ with cut point λ is a-stable if $\lambda \notin \bigcup_{i \in \Sigma} R_i$ for perturbations less than δ,

$$\delta = \min_{i \in \Sigma} \{ | \lambda - R_i | \} \; / \; {}^{o}F \; .$$

<u>Proof</u>: Let x be any tape in Σ^* written in factored form $x = y\,k$ where $k \in \Sigma$ is the last symbol of tape x. Consider the unperturbed and perturbed responses of the automaton to the tape x,

$$rp(x) = \pi_o \, M(y) \, M(k) \, 0_F \qquad \text{(unperturbed)}$$

and

$$rp'(x) = \pi_o \, M'(y) \, M'(k) \, 0_F \qquad \text{(perturbed)} \; .$$

Let $p = [\, p_1 \;\; p_2 \;\; \cdots \;\; p_n]$ and $q = [\, q_1 \;\; q_2 \;\; \cdots \;\; q_n]$ be two stochastic row vectors. Now if δ is the maximum perturbation allowed, i.e., $| M(i) - M'(i) | \leq \delta$ for all $i \in \Sigma$, then we have

$$\sum_{j \in F} \Delta_j^k \leq [\, p_1 \;\; p_2 \;\; \cdots \;\; p_n] \; M(k) \; 0_F \leq \sum_{j \in F} \nabla_j^k$$

and

$$\sum_{j \in F} \Delta_j^k - \delta \cdot {}^{o}F \leq [\, q_1 \;\; q_2 \;\; \cdots \;\; q_n] \; M'(k) \; 0_F \leq \sum_{j \in F} \nabla_j^k + \delta \cdot {}^{o}F \; .$$

Consequently, if the cut point $\lambda \notin \bigcup_{i \in \Sigma} R_i$ then no response point will cross the cut point for perturbations less than

$$\delta = \min_{i \in \Sigma} \{ | \lambda - R_i | \} \; / \; {}^{o}F \; .$$

We now give a simple example to illustrate how this result can be used to yield a-stability results when the automaton is not quasi-definite.

Example: Consider an automaton ρ with cut point $\lambda = 1/2$ defined over the alphabet $\Sigma = \{1, 2\}$ and state set $S = \{1, 2, 3, 4\}$ where $s_o = 1$ is the initial state and $F = \{1, 4\}$ are the designated final states. The given transition matrices are

$$M(1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{4}{5} & \frac{1}{5} \end{bmatrix} \end{array} \qquad M(2) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{6} & 0 & 0 & \frac{5}{6} \\ 0 & \frac{1}{5} & 0 & \frac{4}{5} \\ \frac{1}{10} & 0 & \frac{1}{10} & \frac{4}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} .$$

The response intervals R_i are given by (2.4.1) as

$$R_1 = [0, \tfrac{2}{5}] , \qquad R_2 = [\tfrac{4}{5}, 1] .$$

Now $\lambda = \frac{1}{2} \notin [0, \frac{2}{5}] \cup [\frac{4}{5}, 1]$. Thus, the theorem implies ρ is a-stable for all perturbations less than $\delta = \frac{1}{20}$.

III. ZERO STABILITY PROBLEM

The o-stability problem comes up when one considers the stability of probabilistic automata subjected to small perturbations of the nonzero entries of the symbol matrices. The o-stability problem arises only in nonquasi-definite automata, henceforth denoted NQD automata, since quasi-definite automata are s-stable even under perturbations of zero entries (Theorem 2.3.1). Rabin conjectured that all NQD automata were o-stable, but H. Kesten produced a neat counter example. A slight modification of this example is given below to initiate our study of the problem.

Kesten's Counter Example

Define the NQD probabilistic automaton $K(S, \mathcal{M}, s_1, s_2)$ over the alphabet $\Sigma = \{0, 1\}$ and on the state set $S = \{s_1, s_2, s_3, s_4\}$ by the transition matrices,

$$M(0) = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{bmatrix} p & 1-p & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}, \quad M(1) = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

By induction on n, it is clear that

$$M(0^n) = M^n(0) = \begin{bmatrix} p^n & 1-p^n & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & q^n & 1-q^n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$M(0^n 1) = M(0^n)\, M(1) = \begin{bmatrix} 1-p^n & 0 & p^n & 0 \\ 1 & 0 & 0 & 0 \\ q^n & 0 & 1-q^n & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Now consider the tape $x = (0^n 1)^k \in \Sigma^*$. The transition matrix $M(x)$ is

$$M(x) = \begin{bmatrix} 1-p^n & 0 & p^n & 0 \\ 1 & 0 & 0 & 0 \\ q^n & 0 & 1-q^n & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^k$$

Since the states s_1 and s_3 form a recurrent set with respect to tape $x = (0^n 1)^k$, we consider them separately and determine their limiting behavior as $k \to \infty$. The matrix corresponding to states s_1 and s_3 is denoted by A and is given by

$$A = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} 1-p^n & p^n \\ q^n & 1-q^n \end{bmatrix} \end{array} .$$

The behavior of the limit $Q = \lim_{k \to \infty} A^k$ can easily be determined by expanding A in terms of its constituent matrices as

$$A = U_1 + (1 - p^n - q^n)\, U_2$$

where $\alpha_n = q^n/(p^n + q^n)$ and

$$U_1 = \begin{bmatrix} a_n & 1 - a_n \\ a_n & 1 - a_n \end{bmatrix} , \quad U_2 = I - U_1 \quad .$$

Since $U_1^2 = U_1$, we have $U_1 U_2 = 0$ and $U_2^2 = U_2$. By induction

$$A^k = U_1 + (1 - p^n - q^n)^k U_2 \quad .$$

Thus $\lim_{k \to \infty} A^k = U_1$ for $0 < p, \ q < 1$. Consequently, if we choose $p = q$ in the unperturbed system then we have

$$\lim_{k \to \infty} A^k = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} .$$

Now let us perturb the nonzero entry q to be $q = p - \delta$ where the quantity $\delta > 0$ can be made arbitrarily small. In this case, the perturbed limiting behavior is given by

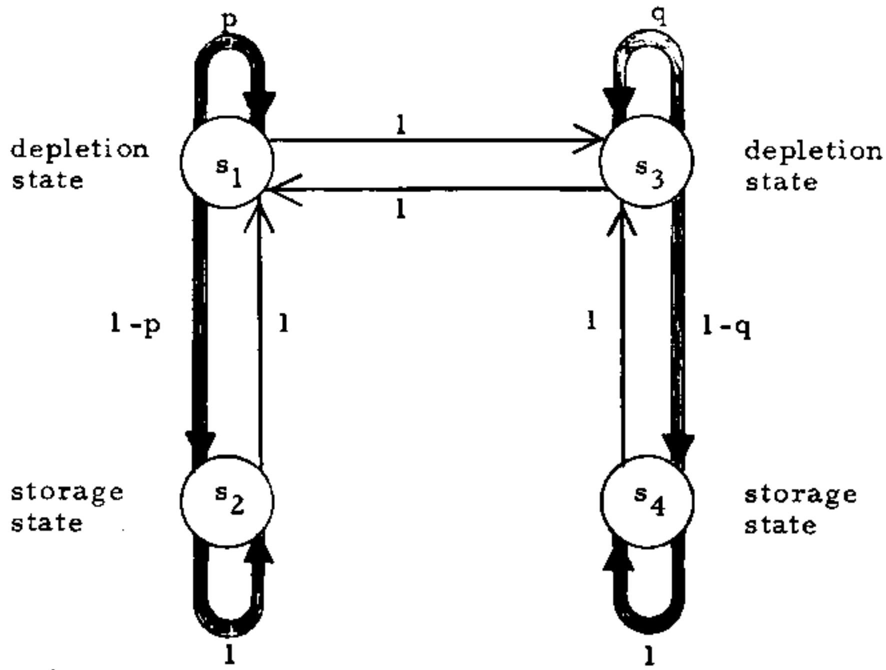
$$\lim_{k \to \infty} A'^k = \begin{bmatrix} a'_n & 1 - a'_n \\ a'_n & 1 - a'_n \end{bmatrix}$$

where $a'_n = (p - \delta)^n / (p^n + (p - \delta)^n)$. Now the quantity a'_n can be made arbitrarily small by choosing n sufficiently large. Consequently, given any $\delta > 0$ and for any $0 < \epsilon < \frac{1}{2}$, there exist integers K and N such that for tape $x = (0^N 1)^K \epsilon \ \Sigma^*$ we have

$$| M(x) - M'(x) | = (\frac{1}{2} - a'_n) > \epsilon \quad .$$

Hence, the NQD automaton K is indeed o-unstable.

The basic idea in this example can be seen by referring to the state diagram of K shown below:



where ▬ and —— denote 0 and 1 transitions respectively.

The transition matrix for tape $x = 0$ is 2-decomposable so that there are two disjoint subsets of states $S^{(1)} = \{s_1, s_2\}$ and $S^{(2)} = \{s_3, s_4\}$ contained in S that are recurrent with respect to tape $x = 0$. Tapes whose transition matrices possess two or more disjoint recurrent subsets are called cycling tapes. The cycling tape $x = 0^n$ is used in the counter example to reduce the probability of being in states s_1 and s_3 to arbitrarily small quantities p^n and q^n respectively. Then, the dump tape $y = 1$ interchanges the probability of being in states s_1 and s_3, and dumps all the probability in s_2 and s_4 back into s_1 and s_3. Now if $p = q$ there is no net transfer of probability between subsets $S^{(1)}$ and $S^{(2)}$. If $q = p - \delta$, $\delta > 0$, however, there is a net transfer of probability from $S^{(1)}$ to

$S^{(2)}$. It is quite clear that if the process is repeated enough times for a given $\delta > 0$ and with a sufficiently large n, essentially all the probability gets transferred to $S^{(2)}$. This causes the o-instability.

The tape $x = (0^n 1)^{2k}$ which revealed the o-instability can be partitioned into "scrambling" subtapes $z = (0^n 1)^2$ with scrambling matrix $M(z)$ as follows:

$$x = z^k = 9)^n 1)^2 \ (0^n 1)^2 \ (0^n 1)^2 \ \ldots \ (0^n 1)^2 \ .$$

This leads us to the important question; what mathematical condition is sufficient to block the quasi-definiteness used in Theorem 2. 3. 1 to prove stability? Clearly, we can write $x = y\, z^k$ and apply the same arguments used in the proof of Theorem 2. 3. 1 to obtain

$$|M(x) - M'(x)| \le \|M(z^k)\| + \|M'(z^k)\| + |M(z^k) - M'(z^k)| \ .$$

However, the minimal nonzero entry α_n in $M(z) = M(0^n \ 1 \ 0^n \ 1)$ is not bounded away from zero as n increases. Hence, we cannot use Theorem 2. 2. 1 on the subtapes z to select an integer K to insure that

$$\|M(z^K)\| \le (1 - \alpha_n)^K \le \epsilon \quad \text{for} \quad 0 < \epsilon < 1 \ .$$

For, assume that one such bound K is found. Clearly, one can choose n sufficiently large for $0 < p, q < 1$ so that $(1 - \alpha_n)^K > \epsilon$, which leads to a contradiction. Thus, we see that the behavior of the minimal nonzero entry attained prior to the scrambling condition plays a fundamental role in zero-stability analysis.

3.1. Cyclic Structure of NQD Automata

Let $\mathcal{G}$ be a NQD automaton defined over the alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_{o_\Sigma}\}$ and on the state set $S = \{s_1, s_2, \ldots, s_n\}$. We now generalize certain notions of a single Markov matrix to apply to our finite family of such matrices. We denote a subset of S by $S^{(i)}$ where i is an integer.

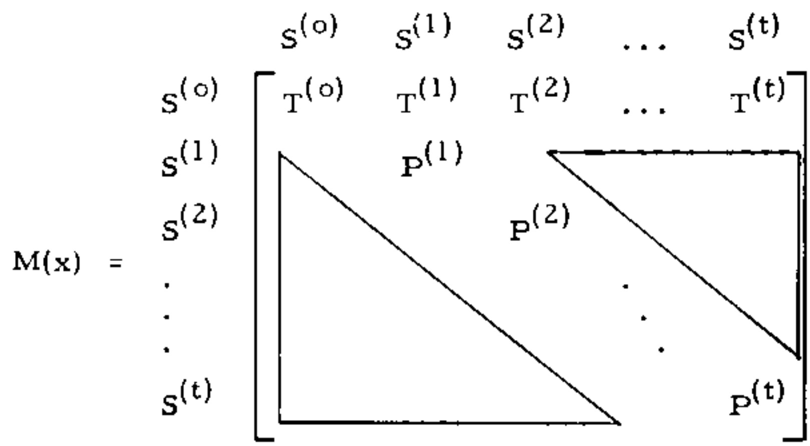
We say that a state s_j is <u>accessible</u> from state s_i by tape $x \in \Sigma^*$, and write $x(s_i) \rightarrow s_j$, if the (i, j) entry in the transition matrix $M(x)$ is nonzero. More generally, we write $x(S^{(i)}) = S_x^{(i)}$ to denote the set of states in S that are accessible from the states in $S^{(i)} \subset S$ by tape x. If for a tape x, $x(S^{(i)}) = S_x^{(i)} \subset S^{(j)}$, then $S^{(i)}$ is mapped by x <u>into</u> $S^{(j)}$, and we denote this by $x(S^{(i)}) \rightarrow S^{(j)}$. An <u>onto</u> mapping is denoted by "$\twoheadrightarrow$".

<u>Definition 3.1.1.</u>: A set of states $S^{(i)} \subset S$ is said to be <u>recurrent</u> with respect to the tape x iff $S_x^{(i)} \subset S^{(i)}$.

<u>Definition 3.1.2.</u>: A tape $x \in \Sigma^*$ is said to be a <u>cycling tape</u> iff there exist two or more disjoint subsets of states in S that are recurrent with respect to x. Each recurrent subset is called a <u>cyclic class</u>.

<u>Definition 3.1.3.</u>: A tape $x \in \Sigma^*$ is said to satisfy the <u>cyclic condition</u> $C(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ iff x is a cycling tape of the cyclic classes $S^{(1)}, S^{(2)}, \ldots, S^{(t)}$. If, in addition, $\bigcup_{i=1}^{t} S^{(i)} = S$, then the cyclic condition is said to be <u>completely t-decomposable</u> (denoted by $C^D(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$).

Clearly, if a tape $x \in \Sigma^*$ satisfies the cyclic condition $C(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ then the transition matrix $M(x)$ is partially t-decomposable. Consequently, Theorem 2.2.4 implies that the matrix $M(x)$ has precisely t unit eigenvalues. Thus, we see the close relationship between the cyclic structure and the number of unit eigenvalues. It is important to note at this point that $C(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ implies only that $x(S^{(i)}) \to S^{(i)}$ for $i = 1, 2, \ldots, t$; and it does not imply that $x(\overline{S}^{(i)}) \to \overline{S}^{(i)}$ where $\overline{S}^{(i)}$ is the complement of $S^{(i)}$ in S. The transition matrix corresponding to a cycling tape x that satisfies $C(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ has the form



upon suitably relabeling the states.

Definition 3.1.4.: A cycling tape $x = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_d}$ is said to be a prime cycling tape if no proper subtape of x of the form $\sigma_{i_j} \sigma_{i_{j+1}} \ldots \sigma_{i_k}$, $1 \leq j \leq k \leq d$, is a cycling tape.

Definition 3.1.5.: A tape x is said to satisfy the prime cyclic condition $C_P(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ iff x is a prime cycling tape of the cyclic classes $S^{(1)}, S^{(2)}, \ldots, S^{(t)}$.

The notion of a prime cycling tape is intimately related to the s-stability problem as we point out in the next theorem.

Theorem 3.1.1.: A probabilistic automaton $\mathcal{P}$ is s-stable iff Σ^* contains no prime cycling tapes.

Proof: (a) If $\mathcal{P}$ is s-stable, then Σ^* contains no cycling tapes. For, if we assume that there exists a prime cycling tape x, then there exists a fixed $\gamma > 0$ such that $\lim_{k\to\infty} \| M(x^k) \| \geq \gamma$. Consequently, s-instability is evident by observing that the symbol matrices of $\mathcal{P}$ can be perturbed so that $\mathcal{P}'$ is quasi-definite, whence $\lim_{k\to\infty} \| M'(x^k) \| = 0$.

(b) If $\mathcal{P}$ is s-unstable then $\mathcal{P}$ is a NQD automaton by Theorem 2.3.1. Thus, for any integer k there exists a tape $x = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_m}$ with $lg(x) = m > k$ for which the matrix $M(x)$ is not scrambling. Consequently, there exists two states s_i and s_j in S such that

$$s_i \xrightarrow{\sigma_{i_1}} S^{(1)} \xrightarrow{\sigma_{i_2}} S^{(2)} \xrightarrow{\sigma_{i_3}} \quad \ldots \quad \xrightarrow{\sigma_{i_m}} S^{(m)}$$

$$s_j \xrightarrow{\sigma_{i_1}} T^{(1)} \xrightarrow{\sigma_{i_2}} T^{(2)} \xrightarrow{\sigma_{i_3}} \quad \ldots \quad \xrightarrow{\sigma_{i_m}} T^{(m)}$$

where $S^{(r)} \cap T^{(r)} = \phi \;\; (r = 1, 2, \ldots, m)$.

Since there are only a bounded number B (see Lemma 3.2.2) of such distinct allocations of the state set S, it follows that any non-scrambling tape x with length $lg(x) > B$ must contain a prime cycling tape.

3.2. Some Properties of the Cyclic Structure

In this section we develop some fundamental properties of the cyclic structure of NQD automata. We now prove some lemmas.

<u>Lemma 3.2.1.</u>: If $x = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_d}$ satisfies $C(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ then for each initial segment $x_1^k = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_k}$ $(k \leq d)$ of x, and for all $1 \leq i, j \leq t$ $(i \neq j)$

$$S^{(i)}_{x_1^k} \cap S^{(j)}_{x_1^k} = \phi \ .$$

<u>Proof:</u> Assume, on the contrary, that there exists two structures $S^{(i)}$ and $S^{(j)}$ of $C(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ such that $S^{(i)}_{x_1^k} \cap S^{(j)}_{x_1^k} = V \neq \phi$. Then

$$\phi \neq V_{x_{k+1}^d} \subset S^{(i)} \cap S^{(j)}$$

in violation of the defining requirement of disjoint cyclic classes of the cyclic condition.

<u>Lemma 3.2.2.</u>: If a tape $x = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_d}$ satisfies a prime cyclic condition $C_p(x; S^{(1)}, S^{(2)})$ then $\lg(x) = d \leq 3^n - 2^{n+1} + 1$, where n is the number of states in S.

<u>Proof:</u> By Lemma 3.2.1, all initial segments x_1^k of x must be such that $S^{(1)}_{x_1^k} \cap S^{(2)}_{x_1^k} = \phi$. The successive pairs of images $S^{(1)}_{x_1^k}$ and $S^{(2)}_{x_1^k}$ are disjoint as illustrated below:

$$S^{(1)} \xrightarrow{\sigma_{i_1}} S^{(1)}_{x_1^1} \xrightarrow{\sigma_{i_1}} S^{(1)}_{x_1^2} \xrightarrow{\sigma_{i_3}} \cdots \xrightarrow{\sigma_{i_d}} S^{(1)}_{x_1^d}$$

$$S^{(2)} \xrightarrow{\sigma_{i_1}} S^{(2)}_{x_1^1} \xrightarrow{\sigma_{i_2}} S^{(2)}_{x_1^2} \xrightarrow{\sigma_{i_3}} \cdots \xrightarrow{\sigma_{i_d}} S^{(2)}_{x_1^d}$$

where $S^{(1)}_{x_1^k} \cap S^{(2)}_{x_1^k} = \phi$ for $k = 1, 2, \ldots, d$. It is important to note that $S^{(1)}_{x_1^k} \cup S^{(2)}_{x_1^k}$ may not be the whole state set. All that is required is that $S^{(1)}_{x_1^k}$ and $S^{(2)}_{x_1^k}$ be nonnull and disjoint. The maximum number of such nonrepetitive, nonnull allocations of n objects can be computed as follows: The number of allocations of subsets of n objects into two sets A and B, so neither A nor B is empty, is equal to the number (3^n) of assignments of n objects to A, B or neither, minus the number (2^n) in which A is empty, minus the number (2^n) in which B is empty, plus the (1) assignment for which both A and B are empty. The total is $d = 3^n - 2 \cdot 2^n + 1$.

3.3. Algorithm for Locating Prime Cyclic Tapes

A bounded algorithm is given here for locating all prime cycling tapes in Σ^*. The algorithm can also be used to decide the quasi-definite condition since, by Theorem 3.1.1, the quasi-definite condition is satisfied if and only if there do not exist any prime cycling tapes. The basic idea involved which stems from Lemmas 3.2.1 and 3.2.2, consists of forming a transition tree of the possible disjoint transitions. More precisely, if $x = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_d}$ satisfies the prime cyclic condition $C_p(x; S^{(1)}, S^{(2)})$, then we have the following sequence of transitions

$$S^{(1)} \xrightarrow{\sigma_{i_1}} S^{(1)}_{x_1^1} \xrightarrow{\sigma_{i_2}} S^{(1)}_{x_1^2} \xrightarrow{\sigma_{i_3}} \cdots \xrightarrow{\sigma_{i_d}} S^{(1)}_{x_1^d} \subset S^{(1)}$$

$$S^{(2)} \xrightarrow{\sigma_{i_1}} S^{(2)}_{x_1^1} \xrightarrow{\sigma_{i_2}} S^{(2)}_{x_1^2} \xrightarrow{\sigma_{i_3}} \cdots \xrightarrow{\sigma_{i_d}} S^{(2)}_{x_1^d} \subset S^{(2)} \quad .$$

Lemma 3.2.1 implies that

$$S^{(1)}_{x_1^k} \cap S^{(2)}_{x_1^k} = \phi \quad \text{for } 1 \le k \le n .$$

Thus, one only needs to consider the possible sequences of disjoint transitions in order to locate the prime cycling tapes in Σ^* .

Let us consider the probabilistic automaton $\mathcal{P}$ defined over the alphabet $\Sigma = \{\sigma_1 , \sigma_2 , \ldots , \sigma_{o\Sigma}\}$ and state set $S = \{1, 2, 3, \ldots, n\}$ by the transition matrices $M(\sigma_i)$, $\sigma_i \in \Sigma$. The algorithm involves constructing a transition tree of the possible state transitions. With each tape $x \in \Sigma^*$, we associate an "access vector" V_x whose i^{th} component $V_x^{(i)}$ is the set of states accessible from state i by tape x . We now construct a transition tree whose vertices are the access vectors and whose directed edges are labeled by the symbols $\sigma_i \in \Sigma$, that map the access vector V_x into the access vector $V_{x\sigma_i}$. The root vertex of the transition tree is the access vector $V_\phi =$ | 1 | 2 | ... | n | corresponding to the null tape ϕ . The transitions emanating from a given vertex are ordered by the symbols σ_i from right to left as $\sigma_1, \sigma_2, \ldots, \sigma_{o_\Sigma}$.

Next, any component $V_x^{(i)}$ and its successors are crossed out if $V_x^{(i)}$ has a nonnull intersection with all other components in V_x. That is to say; we retain only those components $V_x^{(i)}$ of V_x that could generate a cyclic structure.

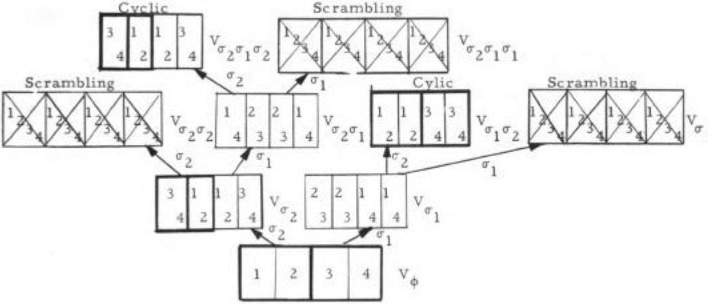
Each branch in the tree is terminated when either 1) all components of the last vertex are crossed out, or 2) there exist two disjoint sets of components that are recurrent from a preceding vertex. The first termination, called a scrambling termination, implies that the scrambling condition has been reached. The second termination, called a cyclic termination, implies that the tape generating the recurrent classes is a prime cycling tape. This algorithm terminates in a bounded number of steps, since the transition tree so defined is bounded by Lemma 3.2.2.

The following examples clarify the essential features encountered in constructing the transition tree.

Example 3.3.1.: Consider the automaton $\mathcal{P}$ defined over the alphabet $\Sigma = \{\sigma_1, \sigma_2\}$ on the state set $S = \{1, 2, 3, 4\}$ by the transition matrices.

$$M(\sigma_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix} \end{array} \quad M(\sigma_2) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array} .$$

The transition tree for the example at hand is given below:



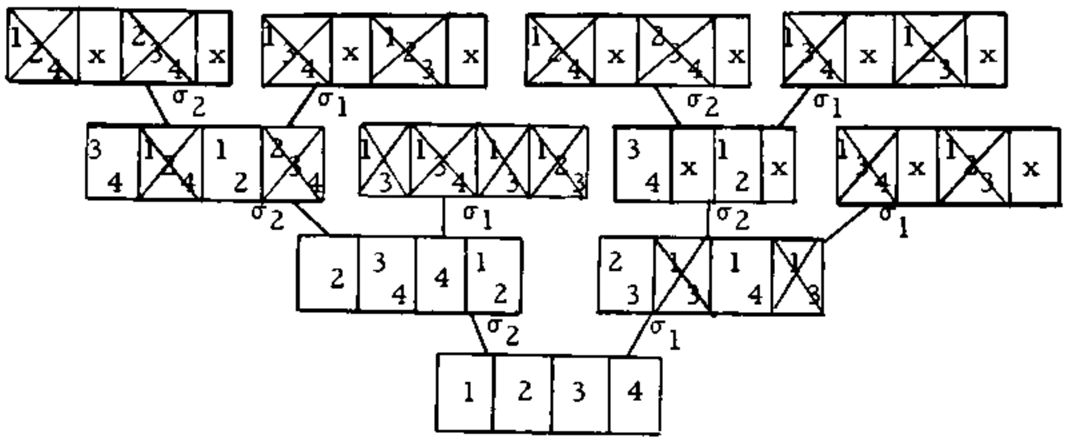
We observe that branch $x = \sigma_1\sigma_2$ possesses two disjoint recurrent subsets, $S^{(1)} = \{1, 2\}$ and $S^{(2)} = \{3, 4\}$ with respect to tape $x = \sigma_1\sigma_2$. That is to say, $S^{(1)}_{\sigma_1\sigma_2} \rightarrow S^{(1)}$ and $S^{(2)}_{\sigma_1\sigma_2} \rightarrow S^{(2)}$. Thus, the tape $x = \sigma_1\sigma_2$ is a prime cycling tape. Also, branch $x = \sigma_2\sigma_1\sigma_2$ again points out that the tape $x = \sigma_1\sigma_2$ is a prime cycling tape. Since all other branches terminate by the scrambling condition, the tape $x = \sigma_1\sigma_2$ is the only prime cycling tape in Σ^*.

Example 3.3.2.: Consider the automaton $\mathcal{G}$ defined over the alphabet $\Sigma = \{\sigma_1, \sigma_2\}$ and on the state set $S = \{1, 2, 3, 4\}$ by the transition matrices,

$$M(\sigma_1) = \begin{bmatrix} 0 & p_1 & 1-p_1 & 0 \\ p_1 & 0 & 1-p_1 & 0 \\ 1-p_1 & 0 & 0 & p_1 \\ 1-p_1 & 0 & p_1 & 0 \end{bmatrix}, \quad M(\sigma_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & p_2 & 1-p_2 \\ 0 & 0 & 0 & 1 \\ p_2 & 1-p_2 & 0 & 0 \end{bmatrix}$$

where $0 < p_1, p_2 < 1$.

The transition tree for this example is given below:



Since all branches in the transition tree are terminated by the scrambling condition, there do not exist any prime cycling tapes in Σ^*. It follows by Theorem 3.1.1, that the given probabilistic automaton is indeed s-stable.

3.4. Quasi-Actual Cyclic Conditions

In this section, the cyclic structure is refined so that we can obtain some o-stability results for NQD automata by methods similar to those used in Theorem 2.3.1. First, we recall that if a tape $x \in \Sigma^*$ satisfies $C(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$, then the transition matrix $M(x)$ is a partially t-decomposable matrix, i.e., $M(x)$ has the form

$$M(x) = \begin{array}{c} \\ S^{(o)} \\ S^{(1)} \\ S^{(2)} \\ \vdots \\ S^{(t)} \end{array} \begin{array}{c} \begin{array}{ccccc} S^{(o)} & S^{(1)} & S^{(2)} & \cdots & S^{(t)} \end{array} \\ \left[\begin{array}{ccccc} T^{(o)} & T^{(1)} & T^{(2)} & \cdots & T^{(t)} \\ & P^{(1)} & & & \\ & & P^{(2)} & & \\ & & & \ddots & \\ & & & & P^{(t)} \end{array} \right] \end{array} \qquad (3.4.1)$$

upon suitably relabeling the states.

Definition 3.4.1.: A partially t-decomposable matrix (3.4.1) is said to be block actual if $T^{(o)} = 0$ and $P^{(i)} > 0$ for $i = 1, 2, \ldots, t$. Similarly, a t-decomposable matrix is said to be block actual iff $P^{(i)} > 0$ for $i = 1, 2, \ldots, t$.

Definition 3.4.2.: A partially t-decomposable matrix (3.4.1) is said to be block quasi-actual iff $T^{(o)} = 0$ and the only zero entries in $P^{(i)}$ occur in columns of zeros.

Definition 3.4.3.: A cyclic condition $C(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ is said to be actual iff the partially t-decomposable matrix $M(x)$ is block actual.

Definition 3.4.4.: A cyclic condition $C(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ is said to be quasi-actual iff the partially t-decomposable matrix $M(x)$ is block quasi-actual.

Definition 3.4.5.: An unbounded product $\prod$ of stochastic matrices P_i is said to be nonlimiting iff there exists a fixed $\alpha > 0$, such that for each positive integer k, the minimal nonzero entry in $\prod_{i=1}^{k} P_i$ is greater than α.

Let A be an $(n \times n)$ stochastic matrix and B be a $(n \times n)$ matrix whose only zero entries occur in columns of zeros. Then for any nonzero entry $c_{i,j}$ in $C = AB$

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} \geq b^{min} \sum_{k=1}^{n} a_{i,k} = b^{min}$$

where b^{min} is the minimal nonzero entry in B. We now have the following lemma.

Lemma 3.4.1.: If each x_i in the tape $x = x_1\, x_2\, x_3 \ldots x_k \ldots$ satisfies a quasi-actual prime cyclic condition $C_p(x; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ then the matrix product $M(x) = M(x_1)\, M(x_2)\, M(x_3) \ldots M(x_k) \ldots$ is nonlimiting.

This result follows from the above inequality by observing that for any positive integer k,

$$\prod_{i=1}^{k} \begin{bmatrix} 0 & T^{(1)}_{x_i} & T^{(2)}_{x_i} & \cdots & T^{(t)}_{x_i} \\ & P^{(1)}_{x_i} & & & \\ & & P^{(2)}_{x_i} & & \\ & & & \ddots & \\ & & & & P^{(t)}_{x_i} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & T^{(1)}_{x_1} \cdot P^{(1)}_{x_2^k} & T^{(2)}_{x_1} \cdot P^{(2)}_{x_2^k} & \cdots & T^{(t)}_{x_1} \cdot P^{(t)}_{x_2^k} \\ & P^{(1)}_{x_1^k} & & & \\ & & P^{(2)}_{x_1^k} & & \\ & & & \ddots & \\ & & & & P^{(t)}_{x_1^k} \end{bmatrix}$$

where $P^{(r)}_{x_i^j} = P^{(r)}_{x_i}\, P^{(r)}_{x_{i+1}} \ldots P^{(r)}_{x_j}$ for $r = 1, 2, \ldots, t$. Throughout this paper, we shall refer to an automaton represented by such a matrix product as a "direct sum" of disjoint automata.

<u>Lemma 3.4.2.</u>: If each prime cycling tape in Σ^* satisfies some actual prime cyclic condition that is completely t-decomposable, then each cycling tape in Σ^* satisfies some quasi-actual cyclic condition with exactly two cyclic classes.

<u>Proof</u>: Consider any cycling tape $x \in \Sigma^*$ that satisifies the cyclic condition $C(x; S^{(1)}, S^{(2)})$. If the tape x is not a prime cycling tape, then x can be partitioned to display a prime cycling tape x_1 as follows:

$$x = y_1 x_1 z_1$$

where y_1 or z_1 could be vacuous. Since x_1 is a prime cycling tape, it satisfies some actual cyclic condition $C_P^D(x_1; T^{(1)}, T^{(2)})$, that is completely decomposable. Consequently, we have for some permutation R of the integers 1 and 2 the following mappings:

$$S^{(o)} \xrightarrow{y_1} S \xrightarrow{x_1} S \xrightarrow{z_1} S^{(1)} \cup S^{(2)}$$

$$S^{(1)} \xrightarrow{y_1} T^{(R(1))} \xrightarrow{x_1} T^{(R(1))} \xrightarrow{z_1} S^{(1)}$$

$$S^{(2)} \xrightarrow{y_1} T^{(R(2))} \xrightarrow{x_1} T^{(R(2))} \xrightarrow{z_1} S^{(2)}$$

where $S^{(o)}$ contains the non-enterable transient states. Since $C_P^D(x_1; T^{(1)}, T^{(2)})$ is actual, for any states $s \in S^{(1)}$ and $t \in S^{(2)}$ we have

$$y_1 x_1(s) = T^{(R(1))} \xrightarrow{z_1} T_{z_1}^{(R(1))} , \quad \forall s \in S^{(1)}$$

and

$$y_1 x_1(t) = T^{(R(2))} \xrightarrow{z_1} T_{z_1}^{(R(2))} , \quad \forall t \in S^{(2)} .$$

Hence, the quasi-actuality condition is preserved.

3.5. Zero-Stability Theorem

In this section we use the cyclic structure to obtain a o-stability result for NQD automata, by methods similar to those used in Theorem 2.3.1. The result extends the Rabin stability problem to NQD automata. We shall omit some of the ϵ and δ details, given in Theorem 2.3.1, that serve only to obscure the initial understanding of the proof.

- A tape $x \in \Sigma^*$ is said to be respectively o-stable, nonlimiting, or scrambling iff the matrix $M(x)$ is o-stable, nonlimiting, or scrambling.

Theorem 3.5.1.: A probabilistic automaton $\mathcal{P}$ is o-stable if each prime cycling tape in Σ^* satisfies some completely 2-decomposable prime cyclic condition that is actual.

Proof: Let x be any tape in Σ^*. First, we shall prove that if $x \in \Sigma^*$ is not a scrambling tape, then x is o-stable and nonlimiting. Secondly, we shall use this fact to prove that all scrambling tapes in Σ^* are o-stable under the conditions of the theorem.

- An ordered collection of disjoint nonempty subsets $S^{(1)}, S^{(2)}, \ldots, S^{(t)}$ of the state set S, written $(S^{(1)}, S^{(2)}, \ldots, S^{(t)})$, is called an t-ordered cyclic structure.

Case 1: If $x \in \Sigma^*$ is not scrambling, then we can partition x into

$$x = y_o x_1 y_1 x_2 y_2 \cdots x_B y_B \tag{3.5.1}$$

where $x_i = \prod_{j=1}^{n_i} x_i^{(j)}$ denotes a concatenation of cycling tapes $x_i^{(j)} \in \Sigma^*$, that satisfy a cyclic condition $C(x_i^{(j)}; S_i^{(1)}, S_i^{(2)})$. The different possible 2-ordered cyclic structures are indexed by i, and the tapes that preserve these ordered cyclic structures by j. The tapes y_i are "transition" tapes that do not contain any cycling subtapes. Clearly, the finite state assumption gives a bound $B = 3^n - 2^{n+1} + 1$ (see Lemma 3.2.2) on the number of distinct 2-ordered cyclic structures, where n is the number of states in S.

For a suitable ordering of the ordered cyclic structures, the concatenated form of x in (3.5.1) is obtained by first segmenting off all cycling tapes $x_1^{(j)}$ as

$$x = y_o x_1 z_1 , \quad x_1 = \prod_{j=1}^{(n_1)} x_1^{(j)} \quad ;$$

so that the tape y_o contains no cycling subtapes, and no initial segment of z_1 satisfies $C(\bullet; S_1^{(1)}, S_1^{(2)})$. Next we segment off all cycling tapes $x_2^{(j)}$ as

$$y = y_o x_1 y_1 x_2 z_2$$

so that the tape y_1 contains no cycling subtapes and no initial segment of z_2 satisfies $C(\bullet; S_2^{(1)}, S_2^{(2)})$. Continuing this process, we obtain the concatenated form of x shown in (3.5.1). In general, the cycling tapes $x_i^{(j)}$ in (3.5.1) are not prime cycling tapes.

Definition 3.4.4.: A tape $x \in \Sigma^*$ is said to generate k ordered cyclic pairs iff k is the minimum integer such that each $x_i^{(j)}$ in the concatenated form (3.5.1) of x,

$$x = y_o x_1 y_1 x_2 y_2 \ldots x_m y_m , \quad \text{where } m \leq k ,$$

generates $(k - 1)$ or fewer ordered cyclic structures. For $k = 0$ we mean that x contains no cycling subtapes.

We now prove that any nonscrambling tape $x \in \Sigma^*$ is o-stable and nonlimiting, using induction on the number of distinct ordered cyclic structures generated by x. First, we show that if $x \in \Sigma^*$ generates only one ordered cyclic structure, then x is o-stable and nonlimiting. We partition x into

$$x = y_o x_1 y_1 \quad ; \quad x_1 = \prod_{j=1}^{n_1} x_1^{(j)} \quad ,$$

where each $x_1^{(j)}$ satisfies $C_P^D(x_1^{(j)}; S_1^{(1)}, S_1^{(2)})$. It follows from Lemma 3.2.2 that the lengths $\lg(y_o)$, $\lg(y_1)$, and $\lg(x_1^{(j)})$ are no greater than $3^n - 2^{n+1} + 1$. We observe that the matrix product $M(x_1)$ is o-stable, since it can be viewed as a direct sum of two disjoint quasi-definite automata (Theorem 2.3.1). Lemma 3.4.1 implies that the matrix product $M(x_1)$ is nonlimiting. Since the lengths $\lg(y_o)$ and $\lg(y_1)$ are no larger than $3^n - 2^{n+1} + 1$, the tape x is o-stable and nonlimiting.

To complete the induction proof, we assume that any nonscrambling tape $x \in \Sigma^*$ that generates k or fewer ordered cyclic structures is o-stable and nonlimiting. Consider any nonscrambling tape $x \in \Sigma^*$ that generates $k + 1$ distinct ordered cyclic structures, and partition it into

$$x = y_o x_1 \, y_1 x_2 \, y_2 \cdots x_m \, y_m \; ; \qquad (m \leq k + 1). \quad (3.5.2)$$

Each cycling tape $x_i^{(j)}$ in (3.5.2) generates k or fewer ordered cyclic structures. Hence, by induction, we know that the matrix products $M(x_i^{(j)})$ in (3.5.2) are o-stable and nonlimiting. That is, given any $\epsilon_1 > 0$ there exists a $\delta(\epsilon_1) > 0$, such that the inequalities

$$|M(\sigma_i) - M'(\sigma_i)| < \delta \qquad \forall \quad \sigma_i \in \Sigma$$

imply (3.5.3)

$$|M(x_i^{(j)}) - M'(x_i^{(j)})| < \epsilon_1$$

for all $x_i^{(j)}$ in (3.5.2). Lemma 3.4.2 implies that each cycling tape $x_i^{(j)} \in \Sigma^*$ satisfies a quasi-actual cyclic condition with two cyclic classes. Note that Lemma 3.4.2 requires each prime cyclic tape to satisfy an actual prime cyclic condition that is completely 2-decomposable. Then, Lemma 3.4.1 implies that each matrix product $M(x_i)$ in (3.5.2) is nonlimiting. Now each matrix product $M(x_i)$ in (3.5.2) is o-stable by Theorem 2.3.1, since each can be viewed as a direct sum of two disjoint quasi-definite automata. Since the length of each y_i in (3.5.2) is bounded by $3^n - 2^{n+1} + 1$ (Lemma 3.2.2), it follows that the matrix product $M(x)$ is o-stable and nonlimiting. That is, given any $\epsilon > 0$ there is a $\epsilon_1 > 0$ and a corresponding $\delta(\epsilon_1) > 0$ in (3.5.3), such that the inequalities

$$|M(\sigma_i) - M'(\sigma_i) < \delta \qquad \forall \quad \sigma_i \in \Sigma$$

imply that

$$|M(x) - M'(x)| < \epsilon \quad .$$

Since the induction is bounded $(k \leq B)$, we conclude that any non-scrambling tape $x \in \Sigma^*$ is indeed o-stable and nonlimiting.

Next, we prove that if $x \in \Sigma^*$ is a scrambling tape, then x is o-stable.

Case 2: If x is a scrambling tape, then it can be partitioned into "prime" scrambling tapes y_i that contain no scrambling subtapes, as follows:

$$x = u w_n ,$$

where $w_n = y_n y_{n-1} \cdots y_2 y_1$. Now, if the length of y_i is no larger than $3^n - 2^{n+1} + 1$, then clearly we can choose our perturbations of the symbol matrices sufficiently small so that $M(y_i)$ is o-stable and nonlimiting. On the other hand, if the length of y_i is greater than $3^n - 2^{n+1} + 1$, then we can delete one symbol and consider $\hat{y}_i$ defined as

$$y_i = \sigma_i \hat{y}_i .$$

Since $\hat{y}_i$ is not scrambling, Case 1 implies that $M(\hat{y}_i)$ is o-stable and nonlimiting. Consequently, the matrix $M(y_i) = M(\sigma_i) M(\hat{y}_i)$ is o-stable and nonlimiting. Hence, the perturbations of the symbol matrices can be chosen sufficiently small so that

$$|M(y_i) - M'(y_i)| < \epsilon_1 \qquad \forall \quad y_i \in x \qquad (3.5.4)$$

for any given $\epsilon_1 > 0$.

Consider the matrix norm

$$|M(x) - M'(x)| = |M(u) M(w_n) - M'(u) M'(w_n)| .$$

By adding and subtracting terms, we get

$$|M(x) - M'(x)| = |(M(u)\, M(w_n) - M(w_n)) - (M'(u)\, M'(w_n) - M'(w_n)) + (M(w_n) - M'(w_n))| \; .$$

The triangle inequality yields

$$|M(x) - M'(x)| \leq |M(u)\, M(w_n) - M(w_n)| + |M'(u)\, M'(w_n) - M'(w_n)| + |M(w_n) - M'(w_n)| \; .$$

Applying Lemma 2.2.4, we obtain

$$|M(x) - M'(x)| \leq \| M(w_n) \| + \| M'(w_n) \| + |M(w_n) - M'(w_n)| \; .$$

Let γ and γ' be the minimal nonzero entries in $\{M(y_i) \mid y_i \in x\}$ and $\{M'(y_i) \mid y_i \in x\}$ respectively. Since each matrix string $M(y_i)$ is nonlimiting, there exists a fixed α such that $\gamma, \gamma' \geq \alpha > 0$. By Theorem 2.2.1, we choose an integer N sufficiently large so that $\| M(w_N) \| \leq (1 - \alpha)^N \leq \frac{\epsilon}{3}$ and $\| M'(w_N) \| \leq (1 - \alpha)^N \leq \frac{\epsilon}{3}$. Since N is a finite integer, we can choose $\epsilon_1 > 0$ in (3.5.4) sufficiently small so that $|M(w_n) - M'(w_n)| < \frac{\epsilon}{3}$. This completes the proof that given any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that the inequalities

$$|M(\sigma_i) - M'(\sigma_i)| < \delta \qquad \forall \quad \sigma_i \in \Sigma$$

imply the inequalities

$$|M(x) - M'(x)| < \epsilon \qquad \forall \quad x \in \Sigma$$

if the zero entries are not perturbed.

3.6. Algebraic Structure of NQD Automata

In this section, we obtain some o-stability results for NQD automata in terms of their algebraic structure. The algebraic systems called semigroups, monoids and groups satisfy, respectively, the first two, three, and four of the following axioms: Let a, b, c represent any elements in a set $\mathcal{S}$; then

A1) Closure Law : $ab \in \mathcal{S}$

A2) Associative Law : $(ab)c = a(bc)$

A3) Identity Law : $\exists\, e_1 \in \mathcal{S}, \ni \forall\, a \in \mathcal{S}$, $a e_1 = e_1 a = a$

A4) Inverse Law : $\forall\, a \in \mathcal{S}, \exists\, a^{-1} \in \mathcal{S} \ni aa^{-1} = a^{-1}a = e_1$

The set of all (n x n) stochastic matrices forms a semigroup. But, although the inverse A^{-1} of an invertible stochastic matrix A has unit row sums, since

$$A^{-1} I_s = A^{-1} AI_s = II_s = I_s ,$$

some entries in A^{-1} will be negative unless A is a permutation matrix. Hence A^{-1} need not be stochastic. For example,

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \quad \text{implies} \quad A^{-1} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} .$$

Thus, one needs an identity element that is more general than the ordinary unit matrix in order to have a potentially fruitful stochastic computing structure.

The general structure of a group of stochastic matrices has been examined by M. Rosenblatt [15], 1965. We shall adopt some of his notation and characterize the structure of certain NQD automata. This will enable us to obtain a o-stability theorem.

<u>Definition 3.6.1.</u>: A $(n \times n)$ stochastic matrix U with identical rows is called a <u>primitive idempotent</u> matrix. It has the form $U = I_s u$ where u is an arbitrary row vector of U.

Since a general idempotent stochastic matrix plays the important role of an identity element in the algebraic structure of probabilistic automata, it is important to determine precisely its structure.

<u>Theorem 3.6.1.</u>: If P is a stochastic idempotent matrix labeled by states, then there is a partitioning of the state set S into disjoint sets of $S^{(o)}, S^{(1)}, \ldots, S^{(t)}$ so that P has the form

$$P = \begin{array}{c} \\ S^{(o)} \\ S^{(1)} \\ S^{(2)} \\ \vdots \\ S^{(t)} \end{array} \begin{array}{c} \begin{array}{ccccc} S^{(o)} & S^{(1)} & S^{(2)} & \cdots & S^{(t)} \end{array} \\ \left[\begin{array}{ccccc} 0 & Q^{(1)}U^{(1)} & Q^{(2)}U^{(2)} & \cdots & Q^{(t)}U^{(t)} \\ & U^{(1)} & & & \\ & & U^{(2)} & & \\ & & & \ddots & \\ & & & & U^{(t)} \end{array} \right] \end{array} \qquad (3.6.1)$$

where $U^{(i)}$ $(i = 1, 2, \ldots, t)$ are positive primitive idempotent matrices. The $Q^{(i)}$ $(i = 1, 2, \ldots, t)$ are ${}^oS^{(o)}$ by ${}^oS^{(i)}$ matrices that can be chosen as

$$Q^{(i)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ {}^{o}S^{(o)} \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \dots & {}^{o}S^{(i)} \end{array} \\ \left[\begin{array}{ccccc} \alpha_1^{(i)} & 0 & 0 & \dots & 0 \\ \alpha_2^{(i)} & 0 & 0 & \dots & 0 \\ \alpha_3^{(i)} & 0 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ \alpha_{{}^{o}S^{(o)}}^{(i)} & 0 & 0 & \dots & 0 \end{array} \right] \end{array} \qquad (3.6.2)$$

where $0 \leq \alpha_j^{(i)} \leq 1$ and $\sum_{i=1}^{t} \alpha_j^{(i)} = 1$ for $j = 1, 2, \dots, {}^{o}S^{(o)}$.

__Proof:__ We recall from matrix theory that P can be transformed into Jordan form Λ by a suitable similarity transformation,

$$S^{-1} P S = \Lambda \ .$$

Since $P^2 = P$, we have

$$S^{-1} P^2 S = S^{-1} P S = \Lambda^2 = \Lambda$$

whence it follows that the eigenvalues of P are either 1 or 0. Theorem 2.2.5 implies that P is partially t-decomposable. That is to say, there is a partitioning of the state set S such that P has the block form

$$P = \begin{array}{c} \\ S^{(o)} \\ S^{(1)} \\ S^{(2)} \\ \vdots \\ S^{(t)} \end{array} \begin{array}{c} \begin{array}{ccccc} S^{(o)} & S^{(1)} & S^{(2)} & \dots & S^{(t)} \end{array} \\ \left[\begin{array}{ccccc} T^{(o)} & T^{(1)} & T^{(2)} & \dots & T^{(t)} \\ & P^{(1)} & & & \\ & & P^{(2)} & & \\ & & & \ddots & \\ & & & & P^{(t)} \end{array} \right] \end{array} \ .$$

Note that, if P had a 0 column under $S^{(i)}$ for $i > o$, then this column could be included in $S^{(o)}$ by a relabeling of states. Thus, each $P^{(i)}$ can be made a positive idempotent matrix that has precisely one unit eigenvalue. Consequently, each $P^{(i)}$ has rank 1. The stochastic condition then implies that each $P^{(i)}$ has identical rows. Hence, each $P^{(i)}$ is a positive primitive idempotent.

The eigenvalues of $T^{(o)}$ are all zero, since $T^{(o)}$ is nilpotent, i.e., there exists an integer m such that $(T^{(o)})^m = 0$. Since $T^{(o)}$ is also idempotent, it follows that $T^{(o)} = (T^{(o)})^m = 0$. The idempotency of P requires each $T^{(i)}$ to satisfy

$$T^{(i)} = T^{(i)} P^{(i)} . \qquad (3.6.3)$$

Each positive primitive idempotent $P^{(i)}$ has the factorization

$$P^{(i)} = I_s^{(i)} p^{(i)}$$

where $p^{(i)}$ is a positive stochastic row vector. If $I_{1,\bullet}^{(i)}$ denotes the first row of the $({}^oS^{(i)} \times {}^oS^{(i)})$ unit matrix $I^{(i)}$, then $I_{1,\bullet}^{(i)} \; I_s^{(i)} = 1$, so

$$T^{(i)} = T^{(i)} P^{(i)} = T^{(i)} I_s^{(i)} (I_{1,\bullet}^{(i)} \; I_s^{(i)}) \, p^{(i)}$$

$$= (T^{(i)} \; I_s^{(i)}) \, I_{1,\bullet}^{(i)} P^{(i)} = Q^{(i)} P^{(i)}$$

where $Q^{(i)}$ is defined in (3.6.2).

It is convenient to employ the concept of o-equivalence of two matrices.

A is o-equivalent to B (denoted $A \overset{o}{\sim} B$) iff
- A and B are $(n \times n)$ matrices having the same zero configuration.

In general, a probabilistic automaton $\mathcal{P}$ whose set of transition matrices $\mathcal{A} = \{ M(x) \mid x \in \Sigma^* \}$ forms a monoid need not be o-stable. In fact, if one designates the identity element U as the ordinary unit matrix, one can construct an example similar to Kesten's, that is also o-unstable. However, if we enrich the "monoid" automaton with the additional property that each element $P \in \mathcal{A}$ has a corresponding "reset element" $P^r \in \mathcal{A}$ such that $P\,P^r \overset{o}{\sim} U$ then we can obtain o-stability. We call such a system a <u>reset monoid</u>. The "reset property" gives the automaton the ability to reset itself back into the idempotent cyclic structure. It is important to notice that the "reset property" does not assume $P\,P^r = U$. However, the following development proves that the reset monoid conditions do imply that $P\,P^r = U$. That is to say, a monoid set $\mathcal{A}$ of $(n \times n)$ stochastic matrices is a group if each element $P \in \mathcal{A}$ has a corresponding reset element $P^r \in \mathcal{A}$ such that $P\,P^r \overset{o}{\sim} U$.

We begin by considering a "zero-reset" probabilistic automaton $\mathcal{P}$ whose set of transition matrices $\mathcal{A} = \{ M(x) \mid x \in \Sigma^* \}$ contains a block actual partially t-decomposable matrix U such that,

$$P\,U \overset{o}{\sim} P \quad \forall \quad P \in \mathcal{A} \quad , \tag{3.6.4}$$

and such that for each $P \in \mathcal{A}$ there is a corresponding "reset element" $P^r \in \mathcal{A}$ satisfying

$$P\,P^r \overset{o}{\sim} U \quad . \tag{3.6.5}$$

By definition, there is a partitioning of the state set so that the matrix U has the form

$$U = \begin{array}{c} \\ S^{(o)} \\ S^{(1)} \\ S^{(2)} \\ \vdots \\ S^{(t)} \end{array} \begin{array}{c} \begin{array}{ccccc} S^{(o)} & S^{(1)} & S^{(2)} & \cdots & S^{(t)} \end{array} \\ \left[\begin{array}{ccccc} 0 & T^{(1)} & T^{(2)} & \cdots & T^{(t)} \\ & B^{(1)} & & & \\ & & B^{(2)} & & \\ & & & \ddots & \\ & & & & B^{(t)} \end{array} \right] \end{array} \tag{3.6.6}$$

where each $B^{(i)} > 0$, i.e., all entries in $B^{(i)}$ are greater than zero. Let P be any matrix in $\mathcal{A}$ and partition the rows and columns as in U,

$$P = \begin{array}{c} \\ S^{(o)} \\ S^{(1)} \\ S^{(2)} \\ \vdots \\ S^{(t)} \end{array} \begin{array}{c} \begin{array}{ccccc} S^{(o)} & S^{(1)} & S^{(2)} & \cdots & S^{(t)} \end{array} \\ \left[\begin{array}{ccccc} P^{(0,0)} & P^{(0,1)} & P^{(0,2)} & \cdots & P^{(0,t)} \\ P^{(1,0)} & P^{(1,1)} & P^{(1,2)} & \cdots & P^{(1,t)} \\ P^{(2,0)} & P^{(2,1)} & P^{(2,2)} & \cdots & P^{(2,t)} \\ \vdots & & & & \vdots \\ P^{(t,0)} & P^{(t,1)} & P^{(t,2)} & \cdots & P^{(t,t)} \end{array} \right] \end{array} . \tag{3.6.7}$$

One readily observes from (3.6.4) that $P^{(i,0)} = 0$ for $i = 0, 1, 2, \ldots, t$. Then (3.6.4) implies

$$P^{(i,j)} \overset{o}{\sim} P^{(i,j)} B^{(j)} . \tag{3.6.8}$$

In order that $P P^{r} \overset{o}{\sim} U$, it is necessary by Lemma 3.2.1 that the matrices P and P^r generate the following mappings:

$$
\begin{array}{ccccc}
S^{(1)} & \xrightarrow{P} & T^{(1)} & \xrightarrow{P^r} & S^{(1)} \\
S^{(2)} & \xrightarrow{P} & T^{(2)} & \xrightarrow{P^r} & S^{(2)} \\
\vdots & & \vdots & & \vdots \\
S^{(t)} & \xrightarrow{P} & T^{(t)} & \xrightarrow{P^r} & S^{(t)}
\end{array}
$$

where the sets $T^{(i)}$ $(i = 1, 2, \ldots, t)$ are pairwise disjoint. Now each element $P \in \mathcal{A}$ must be such that $P \, U \, \mathcal{R} \, P$. Hence

$$
\begin{array}{ccccc}
S^{(1)} & \xrightarrow{P} & T^{(1)} & \xrightarrow{U} & T^{(1)} \\
S^{(2)} & \xrightarrow{P} & T^{(2)} & \xrightarrow{U} & T^{(2)} \\
\vdots & & \vdots & & \vdots \\
S^{(t)} & \xrightarrow{P} & T^{(t)} & \xrightarrow{U} & T^{(t)}
\end{array} .
$$

The structure of U in (3.6.6) requires each $S^{(i)}$ to be some $S^{(j)}$. This implies that each element P performs a permutation on the cyclic structure as follows:

$$
\begin{array}{ccc}
S^{(1)} & \xrightarrow{P} & S^{(R(1))} \\
S^{(2)} & \xrightarrow{P} & S^{(R(2))} \\
\vdots & & \vdots \\
S^{(t)} & \xrightarrow{P} & S^{(R(t))}
\end{array}
$$

where R is a permutation on t integers. We summarize these results with the following theorem.

Theorem 3.6.3.: Let $\mathcal{R}$ be any subgroup of the total permutation group on the integers 1, 2, ..., t . If $\mathcal{P}$ is a zero-reset probabilistic automaton, then any matrix $P \in \mathcal{S} = \{M(x) \mid x \in \Sigma^*\}$ has the form

$$
P = \begin{array}{c} \\ s^{(o)} \\ s^{(1)} \\ s^{(2)} \\ \vdots \\ s^{(t)} \end{array}
\begin{array}{c}
\begin{array}{ccccc} s^{(o)} & s^{(1)} & s^{(2)} & \cdots & s^{(t)} \end{array} \\
\left[\begin{array}{ccccc}
0 & P^{(0,1)} & P^{(0,2)} & \cdots & P^{(0,t)} \\
0 & \delta_{1,R(1)}P^{(1,1)} & \delta_{1,R(2)}P^{(1,2)} & \cdots & \delta_{1,R(t)}P^{(1,t)} \\
0 & & & & \\
\vdots & \vdots & \vdots & & \vdots \\
0 & \delta_{t,R(1)}P^{(t,1)} & \delta_{t,R(2)}P^{(t,2)} & \cdots & \delta_{t,R(t)}P^{(t,t)}
\end{array}\right]
\end{array}
\qquad (3.6.9)
$$

for some $R \in \mathcal{R}$, where each $P^{(i,j)} > 0$ and has unit row sums for $1 \le i, \ j \le t$. The notation implies that each row and column except the first has exactly one non-zero block that is positive.

We now consider a "reset monoid" probabilistic automata $\mathcal{P}$ whose set of transition matrices $\mathcal{S} = \{M(x) \mid x \in \Sigma^*\}$ contains a two-sided identity element $U = U^2 \in \mathcal{S}$ and for each P there is a corresponding reset $P^r \in \mathcal{S}$, such that $P P^r \mathrel{\mathcal{R}} U$. Thus, for each $P \in \mathcal{S}$ we assume that

$$U P = P U = P \qquad (3.6.10)$$

and that there is a $P^r \in \mathcal{S}$ such that

$$P P^r \mathrel{\mathcal{R}} U \ . \qquad (3.6.11)$$

The general structure of the stochastic idempotent matrix U is given by Theorem 3.6.1 as

$$U = \begin{array}{c} \\ S^{(o)} \\ S^{(1)} \\ S^{(2)} \\ \vdots \\ S^{(t)} \end{array} \begin{array}{c} \begin{array}{ccccc} S^{(o)} & S^{(1)} & S^{(2)} & \cdots & S^{(t)} \end{array} \\ \left[\begin{array}{ccccc} 0 & Q^{(1)}U^{(1)} & Q^{(2)}U^{(2)} & \cdots & Q^{(t)}U^{(t)} \\ & U^{(1)} & & & \\ & & U^{(2)} & & \\ & & & \ddots & \\ & & & & U^{(t)} \end{array} \right] \end{array} \qquad (3.6.12)$$

where each $U^{(i)}$ is a positive primitive stochastic idempotent. Let P be any matrix in $\mathcal{A}$ partitioned in the same block form as U. Theorem 3.6.2 implies P has the form given in (3.6.9). Then it follows from (3.6.10) that

$$P^{(i,j)} = P^{(i,j)}\,U^{(j)} = \Delta(i,j)\,U^{(j)} \qquad (3.6.13)$$

where $\Delta(i,j)$ is a $^{o}S^{(i)}$ by $^{o}S^{(j)}$ rectangular matrix whose (k, k) entries are ones and whose other entries are 0. Now, if $P\,P^{r} = \overline{U} \mathcal{R} U$ then $\overline{U}$ has the block form

$$\overline{U} = \begin{array}{c} \\ S^{(o)} \\ S^{(1)} \\ S^{(2)} \\ \vdots \\ S^{(t)} \end{array} \begin{array}{c} \begin{array}{ccccc} S^{(o)} & S^{(1)} & S^{(2)} & \cdots & S^{(t)} \end{array} \\ \left[\begin{array}{ccccc} 0 & T^{(1)} & T^{(2)} & \cdots & T^{(t)} \\ & \overline{U}^{(1)} & & & \\ & & \overline{U}^{(2)} & & \\ & & & \ddots & \\ & & & & \overline{U}^{(t)} \end{array} \right] \end{array} .$$

Since each $U^{(i)}$ in (3.6.12) is a primitive idempotent, we have $\overline{U}^{(i)} U^{(i)} = U^{(i)}$ $(i = 1, 2, \ldots, t)$. Consequently, the right identity condition $\overline{U} = \overline{U}\, U$ implies that $\overline{U}^{(i)} = U^{(i)}$ for $i = 1, 2, \ldots, t$. Then, it follows from the left identity condition $\overline{U} = U\, \overline{U}$ that $T^{(i)} = Q^{(i)} U^{(i)}$ $(i = 1, 2, \ldots, t)$, whence $\overline{U} = U$. Thus, we have shown that any monoid set $\mathcal{S}$ of $(n \times n)$ stochastic matrices which has the "reset property" is a group. We note that it is much easier to recognize the "reset property" than the "inverse property", especially on a computer, where round off errors may obscure an exact inverse.

We summarize these results with the following theorem which is equivalent to a theorem of M. Rosenblatt [15], 1965.

<u>Theorem 3.6.3.</u>: Let $\mathcal{R}$ be any subgroup of the total permutation group on the integers $1, 2, \ldots, t$. If a general element **P** contained in a group G of $(n \times n)$ stochastic matrices is partitioned into the same block structure as U, then P has the form

$$P = \begin{bmatrix} 0 & Q^{(R(1))}\, U^{(1)} & Q^{(R(2))}\, U^{(2)} & \ldots & Q^{(R(t))}\, U^{(t)} \\ 0 & \delta_{1,R(1)} \Delta(1,1) U^{(1)} & \delta_{1,R(2)} \Delta(1,2) U^{(2)} & \ldots & \delta_{1,R(t)} \Delta(1,t) U^{(t)} \\ 0 & \delta_{2,R(2)} \Delta(2,1) U^{(1)} & \delta_{2,R(2)} \Delta(2,2) U^{(2)} & \ldots & \delta_{2,R(t)} \Delta(2,t) U^{(t)} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & \delta_{t,R(1)} \Delta(t,1)\, U^{(1)} & \delta_{t,R(2)} \Delta(t,2)\, U^{(2)} & \ldots & \delta_{t,R(t)} \Delta(t,t)\, U^{(t)} \end{bmatrix} \tag{3.6.14}$$

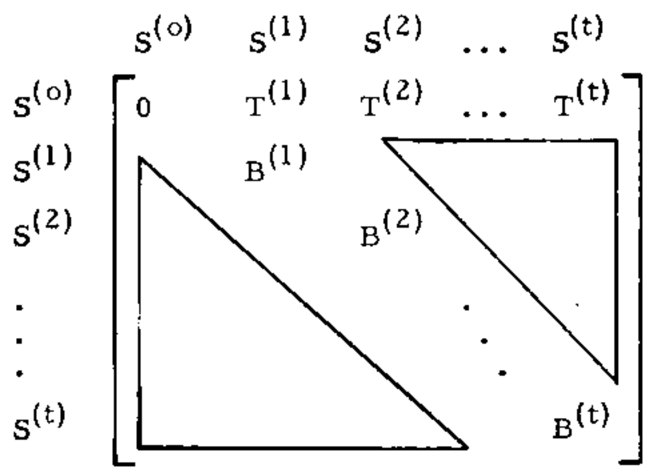
for some R in $\mathcal{R}$, where the $U^{(i)}$ and $Q^{(i)}$ are defined by the stochastic idempotent matrix U in Theorem 3.6.1.

<u>Definition 3.6.2.</u>: A probabilistic automaton $\mathcal{P}$ whose set of stochastic matrices $\mathcal{S} = \{M(x) \mid x \in \Sigma^*\}$ forms a group, is called a <u>group</u> probabilistic automaton.

Our next theorem is a o-stability result for a large class of NQD automata.

<u>Theorem 3.6.4.</u>: Any "zero-reset" probabilistic automaton is o-stable.

<u>Proof</u>: Let $U \in \mathcal{S} = \{M(x) \mid x \in \Sigma^*\}$ denote the given <u>block actual</u> partially t-decomposable matrix. We relabel the states of $\mathcal{P}$ so that U has the form

$U =$ 

where each $B^{(i)} > 0$. Let x be any tape in Σ^*. We partition x, as in Theorem 3.5.1, into cyclic tapes x_i,

$$x = y_o x_1 y_1 x_2 y_2 \cdots x_B y_B$$

where $x_i = \prod_{j=1}^{n_i} x_i^{(j)}$ denotes a product of cycling tapes $x_i^{(j)} \epsilon \Sigma^*$ which satisfy the cyclic condition $C(x_i^{(j)}; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$. The different possible ordered cyclic structures are indexed by i and the tapes which preserve these structures are indexed by j. It follows from Theorem 3.6.2 that there exist no more than $B = t!$ different ordered cyclic structures generated by Σ^*. We now prove by induction on the number of distinct ordered cyclic structures generated by x, that M(x) is o-stable. First, if $x \epsilon \Sigma^*$ generates only one distinct ordered cyclic structure, then it can be partitioned as

$$x = y_o x_1 y_1 , \quad x_1 = \prod_{j=1}^{n_i} x_1^{(j)}$$

where each $x_1^{(j)}$ satisfies an actual cyclic condition $C_P(x_1^{(j)}; S^{(1)}, S^{(2)}, \ldots, S^{(t)})$. From Theorem 3.6.2, it follows that the lengths $lg(y_o)$, $lg(x_1^{(j)})$ and $lg(y_1)$ are no larger than $t!$. It then follows from Theorem 2.3.1 that the matrix product $M(x_1)$ is o-stable, since it can be viewed as a direct sum of t disjoint quasi-definite automata. Since the lengths $lg(y_o)$ and $lg(y_1)$ are no larger than $t!$, we see that M(x) is o-stable.

To complete the induction proof, we assume that any tape $x \epsilon \Sigma^*$ which generates k or fewer distinct ordered cyclic structures is o-stable. Consider a tape $x \epsilon \Sigma^*$ which generates $k + 1$ distinct ordered cyclic structures and partition it as

$$x = y_o x_1 y_1 x_2 \ldots x_{k+1} y_{k+1} . \tag{3.6.15}$$

Now each cyclic tape $x_i^{(j)}$ in (3.6.15) generates k or fewer distinct ordered cyclic structures. Hence, by induction, we know that the matrix products $M(x_i^{(j)})$ in (3.6.15) are o-stable. That is, given any $\epsilon_1 > 0$ there exists a $\delta(\epsilon_1) > 0$ such that the inequalities

$$|M(\sigma_i) - M'(\sigma_i)| < \delta \qquad \forall \ \sigma_i \in \Sigma$$

imply (3.6.16)

$$|M(x_i^{(j)}) - M'(x_i^{(j)})| < \epsilon_1 , \quad \forall \ x_i^{(j)} \in x .$$

Now each matrix string $M(x_i)$ in (3.6.15) is o-stable by Theorem 2.3.1. Since the length of each y_i is bounded by $t!$, it follows that the matrix product $M(x)$ is o-stable. That is, given any $\epsilon > 0$, one can choose $\epsilon_1 > 0$ with corresponding $\delta(\epsilon_1) > 0$ in (3.6.16) sufficiently small that the inequalities

$$|M(\sigma_i) - M'(\sigma_i)| < \delta \qquad \forall \quad \sigma_i \in \Sigma$$

imply that

$$|M(x) - M'(x)| < \epsilon \quad .$$

Since the induction is bounded $(k \leq t!)$, we conclude that any "zero-reset" automaton is o-stable.

We observe in concluding this section that since the group structure clearly implies the "zero-reset" structure, any "group" probabilistic automaton is also o-stable.

IV. ISOLATED CUT-POINT PROBLEM

The concept of an isolated cut point plays an important role in probabilistic automaton theory. The a-stability result of Theorem 2.4.1 and the equivalence between probabilistic automata and deterministic automata depend on the existence of an isolated cut point. As Rabin pointed out in a recent book [2], 1966, there are two open problems in this area. Let $\mathcal{P}$ be a probabilistic automaton with rational cut point λ .

Problem 1: Can one give a procedure for deciding whether or not a given cut point λ is isolated?

Problem 2: Can one give a procedure to determine whether or not $\mathcal{P}$ has any isolated cut points?

The results of this chapter focus on these problems. Our first approach is set theoretic. We define a set of response intervals which contain the response points. Our second approach is topological. In this, we define a pseudo-closure operator that encloses the points which are not isolated cut points.

4.1. Set Theoretic Approach

Consider a probabilistic automaton $\mathcal{P}$ defined over the alphabet $\Sigma = \{1, 2, \ldots, {}^{o}\Sigma\}$ and on the state set $S = \{1, 2, \ldots, n\}$. We begin by giving a very simple sufficient criterion to decide that a given cut point λ is isolated. We follow the development in Section 2.4 by defining the response intervals R_i as in (2.4.2),

$$R_i = [\sum_{j\in F} \Delta^i_j , \sum_{j\in F} \nabla^i_j] \cap [0, 1] \qquad (4.1.1)$$

for all $i \in \Sigma$ where

$$\Delta^i_j = \underset{k}{\text{Min}} \{M(i)_{k,j}\}$$

and

$$\nabla^i_j = \underset{k}{\text{Max}} \{M(i)_{k,j}\} .$$

The range of the response of the j^{th} column of $M(i)$, pre-multiplied by an arbitrary row stochastic vector $p = (p_1, p_2, \ldots, p_n)$, satisfies the inequality

$$\Delta^i_j \le (p_1, p_2, \ldots, p_n) \begin{bmatrix} M^{(i)}_{1,j} \\ M^{(i)}_{2,j} \\ \cdot \\ \cdot \\ \cdot \\ M^{(i)}_{n,j} \end{bmatrix} \le \nabla^i_j . \qquad (4.1.2)$$

Consequently, we have

$$\sum_{j\in F} \Delta^i_j \le \pi_o M(x) M(i) O_F \le \sum_{j\in F} \nabla^i_j \qquad (4.1.3)$$

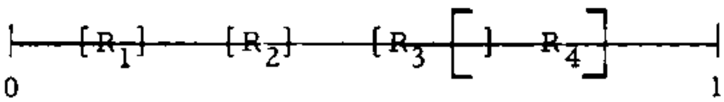
for all $x \in \Sigma^*$.

It is clear from (4.1.3) that all response points are contained in

$$R = \bigcup_{i\in\Sigma} R_i . \qquad (4.1.4)$$

The situation is illustrated on the probability interval, $PI = [0, 1]$ below:

PROBABILITY INTERVAL



where the intervals may overlap. Any points not contained in any R_i are eligible isolated cut points.

Criterion 1: Any cut point $\lambda \in PI - R$ is isolated.

The analysis up to now has given only a sufficient criterion to conclude that a given λ is isolated. It is by no means necessary, since there may exist isolated cut points within the response intervals. The existence of an isolated cut point is guaranteed for actual automata in view of the above results, since the entries in the symbol matrices are positive. However, these extreme cut points may not be very interesting, since they may accept or reject all tapes in Σ^*.

Example 4.1.1.: Consider the actual probabilistic automaton $\mathcal{P}(S, \mathcal{M}, 1, 2)$ defined over the alphabet $\Sigma = \{1, 2\}$ and on the state set $S = \{1, 2\}$ by the symbol matrices

$$M(1) = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}, \quad M(2) = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \end{array} .$$

Since $F = \{2\}$, only the second column need be considered. The response intervals R_i are given by (4.1.1) as

$$R_1 = [\tfrac{1}{4}, \tfrac{1}{2}] \quad \text{and} \quad R_2 = [\tfrac{2}{3}, \tfrac{3}{4}] .$$

By Criterion 1, any cut point $\lambda \in [0, \frac{1}{4}) \cup (\frac{1}{2}, \frac{2}{3}) \cup (\frac{3}{4}, 1]$ is isolated.

We now extend these response intervals to tapes of length N by defining for each tape $x \in \Sigma_N = \{x \mid lg(x) = N, x \in \Sigma^*\}$ a response interval R_x as,

$$R_x = [\sum_{j \in F} \Delta_j^x, \sum_{j \in F} \nabla_j^x] \cap [0, 1] \qquad (4.1.5)$$

where

$$\Delta_j^x = \underset{k}{\text{Min}} \{M(x)_{k,j}\}$$

and

$$\nabla_j^x = \underset{k}{\text{Max}} \{M(x)_{k,j}\} \quad .$$

Now consider any tape x in Σ^* with $lg(x) \geq N$. If we partition x as $x = y z$ where $z \in \Sigma_N$, then it follows from (4.1.3) that $rp(x)$ is contained in R_z. In general, if

$$R^N = \bigcup_{y \in \Sigma_N} R_y \quad ,$$

then the response to any tape $x \in \Sigma^*$ with $lg(x) \geq N$ is contained in R^N. However, the response of tapes whose lengths are less than N may not be contained in R^N.

<u>Criterion 2</u>: Any cut point $\lambda \in PI - R^N$ such that $\lambda \notin \{rp(x) \mid x \in \Sigma^{N-1}\}$ is an eligible isolated cut point.

<u>Example 4.1.2.</u>: Consider the probabilistic automaton $\mathcal{P}(S, \mathcal{M}, 1, 2)$ defined over the alphabet $\Sigma = \{1, 2\}$ and on the state set $S = \{1, 2\}$ by the symbol matrices,

$$M(1) = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}, \quad M(2) = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \end{array}$$

By Criterion 1, any cut point $\lambda \in [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$ is isolated. However, these points are trivial, since they either accept or reject all tapes in Σ^*. Let us now consider all the matrix products of length two:

$$M(11) = \begin{bmatrix} \frac{11}{16} & \frac{5}{16} \\ \frac{5}{8} & \frac{3}{8} \end{bmatrix}, \quad M(21) = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{9}{16} & \frac{7}{16} \end{bmatrix},$$

$$M(12) = \begin{bmatrix} \frac{7}{16} & \frac{9}{16} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}, \quad M(22) = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix}.$$

By Criterion 2, any $\lambda \in [0, \frac{5}{16}) \cup (\frac{7}{16}, \frac{9}{16}) \cup (\frac{11}{16}, 1]$, not a response point $\frac{1}{4}$ or $\frac{1}{2}$, is an eligible isolated cut point. One should note that the response intervals have separated and allow nontrivial isolated cut points within $(\frac{7}{16}, \frac{9}{16})$.

We now prove a theorem which focuses on the isolated cut point problems for quasi-definite automata.

<u>Theorem 4.1.1.</u>: Let $\mathcal{P}$ be a quasi-definite probabilistic automaton defined over the alphabet $\Sigma = \{1, 2, \ldots, {}^{o}\Sigma\}$ and on the state set $S = \{1, 2, \ldots, n\}$. Given any rational cut point λ, one can

conclude with a bounded experiment that either λ is not γ-isolated or λ is γ^*-isolated for any fixed $\gamma > 0$ and some $\gamma^* > 0$.

Proof: Given any $\epsilon > 0$, there is, by the quasi-definite condition, an integer $N(\epsilon)$ such that $lg(x) \geq N$ and $x \in \Sigma^*$ imply that $\| M(x) \| < \epsilon$. A particular integer N can be determined for any $\epsilon > 0$ by Theorem 2.2.1. Let us consider any tape x in Σ^* with $lg(x) > N$. Partition x as $x = y\,z$ where $lg(z) = N$. The quasi-definite condition implies that $\| M(z) \| < \epsilon$. We now write the matrix $M(z)$ as a sum

$$M(z) = U_z + N_z$$

where U_z is a primitive idempotent matrix whose equal rows are the average of the rows of $M(z)$ and where N_z is such that $| N_z | < \epsilon$. Now let P be any $(n \times n)$ stochastic matrix and consider

$$PM(z) = P(U_z + N_z) = PU_z + PN_z = U_z + PN_z .$$

The stochastic condition implies that $| PN_z | \leq | N_z | < \epsilon$. Thus, if we choose $\epsilon = \gamma/(2\ {}^{o}F)$ then the response $rp(x)$ is contained in the interval $R_z^* = [\pi_o U_z O_F - \frac{\gamma}{2},\ \pi_o U_z O_F + \frac{\gamma}{2}]$. Hence, if we wrap a closed γ-neighborhood $\overline{N}(rp(z), \gamma)$ about $rp(z)$, then we enclose the response of $rp(y\,z)$ for all $y \in \Sigma^*$. Consequently, if we form

$$R = \bigcup_{x \in \Sigma^N} \overline{N}(rp(x), \gamma) ,$$

then we can say that any point $\lambda \in R$ is not γ-isolated. On the other hand, if $\lambda \in PI - R$ then we can say that λ is γ^*-isolated, where γ^* is the minimal distance from λ to R.

We complete this section by giving a sufficient condition to conclude, for any quasi-definite automaton $\mathcal{P}$ with a single starting state s_o, and a single final state s_F, that $\mathcal{P}$ has no isolated cut points. Let V be any arbitrary $(n \times 1)$ vector whose components v_i satisfy $0 \leq v_i \leq 1$. We shall call such a vector a probability vector. We define the range of V, $\mathcal{R}(V)$ by

$$\mathcal{R}(V) = [v_{min}, v_{max}]$$

where v_{min} and v_{max} are the minimal and maximal components of V. We say that the symbol matrix $M(\sigma_i)$ covers V if

$$\mathcal{R}(M(\sigma_i)\, V) \supset [v_{min}, v_{max}] \ .$$

More generally, we say that the set of matrices $M(\sigma_i)$, $\sigma_i \in \Sigma$, covers V if

$$\mathcal{R}(V) \subset \bigcup_{\sigma_i \in \Sigma} \mathcal{R}(M(\sigma_i)\, V) \ .$$

Theorem 4.1.2.: Let $\mathcal{P}(S, \mathcal{M}, s_o, s_F)$ be a quasi-definite probabilistic automaton defined over the alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_{o_\Sigma}\}$ and on the state set $S = \{s_1, s_2, \ldots, s_n\}$. If the symbol matrices $M(\sigma_i)$, $\sigma_i \in \Sigma$ cover any probability vector V, then $\mathcal{P}$ has no isolated cut points.

Proof: We observe for any tape $z \in \Sigma_N$ that if the probability vector V is chosen as the column of $M(z)$ corresponding to the final state s_F, then the range, $\mathcal{R}(V)$, of V is identical to the response interval, R_z, defined in (4.1.5). One can easily show using induction on the tape length that

$$R^N = \bigcup_{z \in \Sigma_N} R_z = [0, 1] \qquad \forall \; N ,$$

since the symbol matrices cover any probability vector. The quasi-definite condition requires the length $L(R_z)$ of each response interval R_z to approach 0 as lg(z) becomes infinite. Since each response interval contains at least one response point, it follows that $\mathcal{P}$ has no isolated cut points.

<u>Example 4.1.3.</u>: Consider the quasi-definite probabilistic automaton $\mathcal{P}$ (S, $\mathcal{M}$, s_1, s_2) defined over the alphabet $\Sigma = \{0, 1\}$ and on the state set $S = \{s_1, s_2\}$ by the symbol matrices

$$M(0) = \begin{bmatrix} 1 & 0 \\ a & 1-a \end{bmatrix}, \quad M(1) = \begin{bmatrix} 0 & 1 \\ a & 1-a \end{bmatrix} .$$

We apply the theorem to show that $\mathcal{P}$ has no isolated cut points if $0 < a < 1$.

Let $V = (v_1, v_2)^T$ be any probability vector with $v_1 \leq v_2$, and consider the ranges

$$\mathcal{R}(M(0)\, V) = [\, v_1, \; a v_1 + (1-a)\, v_2]$$

and

$$\mathcal{R}(M(1)\, V) = [\, a v_1 + (1-a)\, v_2, \; v_2]$$

which clearly covers V. Similarly, if $v_1 > v_2$ then these symbol matrices again cover V. Consequently, the theorem implies that $\mathcal{P}$ has no isolated cut points.

4.2. Topological Approach

The topological approach of this section gives a neat view of the tape acceptance behavior of probabilistic automata. The isolated cut-point problem is viewed in a more enlightening setting. The results of this section focus on the isolated cut point problems for an arbitrary probabilistic automaton.

Response Model

Let $\mathcal{P}$ be an arbitrary probabilistic automaton defined over the alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_{o_\Sigma}\}$ and on the state set $S = \{s_1, s_2, \ldots, s_n\}$. We introduce a "response model" $\mathcal{R}$ for $\mathcal{P}$ to characterize the response of all tapes in Σ^* in terms of the response of tapes in Σ^{n-1}. The central idea, which stemmed from a recent paper by J. W. Carlyle [3], brings into consideration the constraints imposed by the finite state assumption. We recall that the response of $\mathcal{P}$ to a tape $x \in \Sigma^*$ is defined by the bilinear form

$$rp(x) = \pi_o M(x) O_F .$$

Consider any collection of $2m$ $(m > n)$ tapes

$$x_1, x_2, \ldots, x_m \quad \text{and} \quad \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$$

from Σ^*. Collect the response of $\mathcal{P}$ to these tapes into a "response matrix" P defined as follows:

$$P = \begin{array}{c} \\ x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{array} \begin{array}{c} \begin{array}{cccc} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_m \end{array} \\ \begin{bmatrix} rp(x_1\bar{x}_1) & rp(x_1\bar{x}_2) & \cdots & rp(x_1\bar{x}_m) \\ rp(x_2\bar{x}_1) & rp(x_2\bar{x}_2) & \cdots & rp(x_2\bar{x}_m) \\ \vdots & & & \vdots \\ rp(x_m\bar{x}_1) & rp(x_m\bar{x}_2) & \cdots & rp(x_m\bar{x}_m) \end{bmatrix} \end{array} \qquad (4.2.1)$$

The response matrix P has the following factorization,

$$P = Q\,H \qquad (4.2.2)$$

where Q and H are $(m \times n)$ and $(n \times m)$ matrices respectively defined by

$$Q = \{ Q_{i,\cdot} \mid Q_{i,\cdot} = \pi_o M(x_i) \}$$

and

$$H = \{ H_{\cdot,j} \mid H_{\cdot,j} = M(\bar{x}_j)\, O_F \} \quad .$$

It follows from the factorization (4.2.2) that the rank of the response matrix P is no larger than n. Hence, the determinant of P, denoted by $\det(P)$; is zero, independent of the tapes selected.

<u>Definition 4.2.1.</u>: The <u>rank</u> of the response model $\mathcal{R}$ is defined to be the largest rank $r\ (\leq n)$ of the response matrix P that can be generated from Σ^{n-1}.* If $r = n$ then $\mathcal{R}$ is said to have maximum rank.

Now consider one collection of $2r$ tapes

$$x_1,\ x_2,\ \ldots,\ x_r \quad \text{and} \quad \bar{x}_1,\ \bar{x}_2,\ \ldots,\ \bar{x}_r$$

*If follows from a theorem of Carlyle [3] that the tapes in Σ^{n-1} determine the rank of the response model.

in Σ^{n-1} such that the rank of the response matrix P is r. Let k and $\overline{k}$ be defined as

$$k = \underset{1 \leq i \leq r}{\mathrm{Max}} \{ \lg(x_i) \}$$

and

$$\overline{k} = \underset{1 \leq i \leq r}{\mathrm{Max}} \{ \lg(\overline{x}_i) \} \quad .$$

Consider two tapes $x \in \Sigma_{k+1}$ and $\overline{x} \in \Sigma_{\overline{k}+1}$ and form the response matrix P partitioned as

$$P = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_r \\ x \end{array} \begin{array}{c} \begin{array}{ccccc} \overline{x}_1 & \overline{x}_2 & \cdots & \overline{x}_r & \overline{x} \end{array} \\ \left[\begin{array}{cccc|c} rp(x_1\overline{x}_1) & rp(x_1\overline{x}_2) & \cdots & rp(x_1\overline{x}_r) & rp(x_1\overline{x}) \\ rp(x_2\overline{x}_1) & rp(x_2\overline{x}_2) & \cdots & rp(x_2\overline{x}_r) & rp(x_2\overline{x}) \\ \vdots & & & & \\ rp(x_r\overline{x}_1) & rp(x_r\overline{x}_2) & \cdots & rp(x_r\overline{x}_r) & rp(x_r\overline{x}) \\ \hline rp(x\,\overline{x}_1) & rp(x\,\overline{x}_2) & \cdots & rp(x\,\overline{x}_r) & rp(x\,\overline{x}) \end{array}\right] \end{array} \tag{4.2.3}$$

Label the partitioned response matrix P as

$$P = \begin{bmatrix} A & B(\overline{x}) \\ C(x) & D \end{bmatrix} \quad .$$

We then observe that the response matrix P can be factored as

$$P = \begin{bmatrix} A & B(\overline{x}) \\ C(x) & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ C(x)A^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & (D-C(x)A^{-1}B(\overline{x})) \end{bmatrix} \begin{bmatrix} I & A^{-1}B(\overline{x}) \\ 0 & I \end{bmatrix}$$

since A^{-1} exists. Hence, we have

$$\det(P) = \det(A) \cdot \det(D - C(x)A^{-1}B(\bar{x})) = 0 .$$

Since D is a scalar and $\det(A) \neq 0$, we have

$$D = rp(x\bar{x}) = C(x)A^{-1}B(\bar{x}) \qquad (4.2.4)$$

where $lg(x\bar{x}) = k + \bar{k} + 2$. Thus, the response of tapes in $\Sigma_{k+\bar{k}+2}$ can be determined in terms of the response of tapes in $\Sigma^{k+\bar{k}+1}$ by (4.2.4).

We define Y_N to be a set of response points, one for each tape in Σ_N,

$$Y_N = \{rp(x) \mid x \in \Sigma_N\} .$$

In a similar manner Y^N denotes the set of response points for the tapes in Σ^N,

$$Y^N = \{rp(x) \mid x \in \Sigma^N\} .$$

The bilinear form in (4.2.4) implies that $Y_{k+\bar{k}+2}$ can be determined from $Y^{k+\bar{k}+1}$. In general, the linear response transformation $C(x)A^{-1}$ of (4.2.4) may be considered fixed by determining $C(x)A^{-1}$ for each tape $x \in \Sigma_{k+1}$. This yields a finite family of response (row) vectors, $\mathcal{R} = \{C(x)A^{-1} \mid x \in \Sigma_{k+1}\}$, one for each tape in Σ_{k+1}, called the <u>response model</u>. The response model, $\mathcal{R} : Y^N \to Y_{N+1}$, for $N \geq k + \bar{k} + 1$; defines the entire tape response of $\mathcal{P}$ recursively by

$$Y_{N+1} = \{R_x B(\bar{x}) \mid R_x \in \mathcal{R} , \bar{x} \in \Sigma_{N-k}\}$$

for $N \geq k + \bar{k} + 1 = L$. We note that this response model can be used efficiently to carry out the decision algorithm of Theorem 4.1.1.

<u>Theorem 4.2.1.</u>: Each response vector R_x contained in the response model $\mathcal{R}$ has unit row sum.

<u>Proof</u>: Consider the response matrix P partitioned as shown in (4.2.3). Each response vector R_x contained in $\mathcal{R}$ is defined by

$$R_x = C(x) A^{-1}$$

which implies that

$$C(x) = R_x A \ .$$

Consequently, if $R_x = [r_1 \ r_2 \ \ldots \ r_r]$ then we have

$$(r_1 \pi_o M(x_1) + r_2 \pi_o M(x_2) + \ldots + r_r \pi_o M(x_r) - \pi_o M(x)) M(x_i) O_F = 0$$

for each x_i $(i = 1, 2, \ldots, r)$. Since A^{-1} exists, it follows that the vectors $\pi_o M(x_i)$ $(i = 1, 2, \ldots, r)$ are linearly independent. This implies that there exists a unique vector $R_x = [r_1 \ r_2 \ \ldots \ r_r]$ such that

$$r_1 \pi_o M(x_1) + r_2 \pi_o M(x_2) + \ldots + r_r \pi_o M(x_r) = \pi_o M(x) \ .$$

This in turn implies that R_x satisfies $C(x) = R_x A$. Since the vectors $\pi_o M(x_i)$ $(i = 1, 2, \ldots, r)$ are stochastic, it follows that R_x must have unit row sum. It may have, however, some negative entries.

A simple example is given here to illustrate the essential features of the response model.

<u>Example 4.2.1.</u>: Consider a probabilistic automaton $\mathcal{P}(S, \mathcal{M}, s_1, s_2)$ defined over the alphabet $\Sigma = \{0, 1\}$ and on the state set $S = \{s_1, s_2\}$ by the symbol matrices,

$$M(0) = \begin{bmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}, \quad M(1) = \begin{bmatrix} 0 & 1 \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

Choose $x_1 = \bar{x}_1 = 0$ and $x_2 = \bar{x}_2 = 1$. The response matrix $\mathbf{P}$ in (4.2.1) is defined for tapes x and $\bar{x}$ in Σ_2 as

$$P = \begin{array}{c} \\ 0 \\ 1 \\ x \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & \bar{x} \end{array} \\ \begin{bmatrix} 0 & 1 & rp(0\,\bar{x}) \\ \frac{1}{4} & \frac{1}{4} & rp(1\,\bar{x}) \\ rp(x\,0) & rp(x\,1) & rp(x\,\bar{x}) \end{bmatrix} \end{array}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad C(x) = [\, rp(x\,0) \;\; rp(x\,1)]$$

and $B^T(x) = [\, rp(0\,\bar{x}) \;\; rp(1\,\bar{x})]$. The response model $\mathcal{R}$ is formed by determining for each tape $x \in \Sigma_2$, $R_x = C(x)A^{-1}$ as follows:

$$\begin{aligned} R_{00} &= C(00)A^{-1} = [\,1 \;\; 0] \\ R_{01} &= C(01)A^{-1} = [\,0 \;\; 1] \\ R_{10} &= C(10)A^{-1} = [\,\tfrac{3}{4} \;\; \tfrac{1}{4}] \\ R_{11} &= C(11)A^{-1} = [\,\tfrac{3}{4} \;\; \tfrac{1}{4}] \, . \end{aligned}$$

The entire tape response of $\mathcal{P}$ is given recursively by

$$Y_{N+1} = \{ R_x B(\bar{x}) \mid R_x \in \mathcal{R} , \bar{x} \in \Sigma_{N-1} \}$$

for all $N \geq 3$ where $rp(x \bar{x}) = R_x B(\bar{x})$.

The response model for this example can be organized into the following convenient matrix recurrence equation. Define the partitioned matrices

$$P_3 = [B(00), B(01), B(10), B(11)] ,$$

$$C_1 = \begin{bmatrix} C(00)A^{-1} \\ C(10)A^{-1} \end{bmatrix} , \text{ and}$$

$$C_2 = \begin{bmatrix} C(01)A^{-1} \\ C(11)A^{-1} \end{bmatrix} .$$

Now if $P_4 = C_1 P_3$ and $P'_4 = C_2 P_3$, then the response model is given by

$$C_1[P_i \quad P'_i] = P_{i+1}$$

$$C_2[P_i \quad P'_i] = P'_{i+1} \qquad (i = 4, 5, \ldots)$$

where the matrices P_{i+1} and P'_{i+1} contain the response points for tapes in Σ_{i+1} . The notation $[P_i \quad P'_i]$ denotes a partitioned matrix formed from P_i and P'_i .

Pseudo-Closure Operator

The setting for the following development is the probability interval PI = [0, 1] . Let X_k denote a set of closed γ-neighborhoods $\overline{N}(\cdot, \gamma)$, one around each response point in Y_k,

$$X_k = \{ \overline{N}(rp(x), \gamma) : x \in \Sigma_k \} .$$

Similarly, let X^k denote a set of closed γ-neighborhoods $\overline{N}(\cdot, \gamma)$, one around each response point in Y^k

$$X^k = \{\overline{N}(rp(x), \gamma) : x \in \Sigma^k \} .$$

The "response operator" $C_\gamma : Y^k \rightarrow X^{k+1}$, is defined by

$$C_\gamma(Y^k) = C_\gamma^{(o)}(Y^{(k)}) \cup \overline{N}(\mathcal{R}(Y^k), \gamma) = X^{k+1}$$

for all $k \geq L = k + \overline{k} + 1$ where $C_\gamma^{(o)}(Y^k) = X^k$ and $\overline{N}(\mathcal{R}(Y^k), \gamma) = X_{k+1}$. The important point to notice here is that the response operator utilizes the response model $\mathcal{R}$ to generate Y_{k+1} from Y^k. It then wraps γ-neighborhoods about the points in Y_{k+1} while retaining all previously generated neighborhoods. The composite response operator is defined recursively by

$$C_\gamma^k (Y^L) = C_\gamma^{k-1}(Y^{L+1}) = C_\gamma^{k-2}(Y^{L+2}) = \ldots = C(Y^{L+k-1}) .$$

Consequently, the response operator has the following important nesting property,

$$C_\gamma^{(o)}(Y^L) \subset C_\gamma^{(1)}(Y^L) \subset \ldots \subset C_\gamma^{(k-1)}(Y^L) \subset C_\gamma^{(k)}(Y^L) \subset \ldots .$$

The following discussion centers on the response operator and pertains to the solvability of the isolated cut-point problems for an arbitrary probabilistic automaton. Let C_{γ} and $C_{\gamma+\epsilon}$ denote response operators that enclose the response points with γ and $\gamma+\epsilon$ closed neighborhoods respectively, where ϵ is any fixed positive number. We shall prove that if one can effectively decide whether or not $C_{\gamma}^{n}(Y^{L}) \subset C_{\gamma+\epsilon}^{N}(Y^{L})$ for all $n \geq N$, then the isolated cut-point problems are recursively solvable. A cursory examination of the continuity of the linear response model indicates that such an effective procedure does in fact exist. If this is true, then the following conjecture would follow:

<u>Conjecture 2.4.2.</u>: For an arbitrary probabilistic automaton, there exists a finite integer N such that one can decide from $C_{\gamma+\epsilon}^{N}(Y^{L})$ that a given cut point λ is either γ-isolated or not $(\gamma+\epsilon)$-isolated for any $\epsilon > 0$.

The following discussion pertains to the above conjecture.

<u>Case 1</u>: We first observe that if there exists an integer N such that $C_{\gamma}^{n}(Y^{L}) \subset C_{\gamma}^{N}(Y^{L})$ for all $n \geq N$ then we can decide that

a) $\lambda \in C_{\gamma}^{N}(Y^{L})$ is not γ-isolated

b) $\lambda \in PI - C_{\gamma}^{N}(Y^{L})$ is γ-isolated.

<u>Case 2</u>: If no finite integer N exists such that $C_{\gamma}^{n}(Y^{L}) \subset C_{\gamma}^{N}(Y^{L})$ for $n \geq N$, then consider the set

$$B_{n} = PI - C_{\gamma}^{n}(Y^{L}) .$$

Let ϵ_{1} be any fixed positive number. If on the one hand, there exists for all n an integer N_1 such that the measure of $B_n - B_{n+N_1}$ satisfies $m(B_n - B_{n+N_1}) \geq \epsilon_1$, then there is a finite integer N such that $B_N = \phi$. Consequently, the automaton has no γ-isolated cut points. If on the other hand, no integer N_1 exists such that $m(B_n - B_{n+N_1}) \geq \epsilon_1$ then we have a limiting situation. Let C_γ and $C_{\gamma+\epsilon}$ denote response operators that enclose the response points with γ and $\gamma+\epsilon$ closed neighborhoods respectively. Now stop the closure process when

$$C_\gamma^n(Y^L) \subset C_{\gamma+\epsilon}^N(Y^L) \quad \text{for all } n \geq N .$$

This will be true for some finite integer N, since no integer N_1 exists such that $m(B_n - B_{n+N_1}) \geq \epsilon$ for any $\epsilon > 0$. In this case, one can decide that

a) $\lambda \notin C_{\gamma+\epsilon}^N(Y^L)$ is γ-isolated

b) $\lambda \in C_{\gamma+\epsilon}^N(Y^L)$ is not $(\gamma+\epsilon)$-isolated.

V. CONCLUSIONS

In this chapter we summarize the important original results obtained. Chapter 2 contains a necessary and sufficient condition for strict stability (Theorem 2.3.1). A bounded algorithm is given in Section 3.3 which efficiently solves the strict stability problem for an arbitrary probabilistic automaton. The algorithm is particularly suited for a digital computer. It requires only logical operations and does not require the multiplication of matrices. Theorem 2.4.2 gives a sufficient condition for tape acceptance stability without requiring the automaton to be strictly stable. This result is essentially a regional stability result, since it gives a bound on the size of perturbations that can be permitted without causing tape acceptance instability.

Chapter 3 contains zero-stability results for NQD automata in terms of their cyclic and algebraic structures. Section 3.2 contains some fundamental properties of the cyclic structure of NQD automata. These results led to the algorithm given in Section 3.3 for locating all prime cycling tapes for an arbitrary probabilistic automaton. Section 3.4 refines the cyclic structure, so that the minimal nonzero entry in a matrix product does not approach zero prior to attaining the scrambling condition. Theorem 3.5.1 gives sufficient conditions for zero-stability in terms of the prime cyclic conditions. The method of proof is similar to that of Theorem 2.3.1. The crucial point is to prevent the minimal nonzero entry from having a zero limit prior to attaining the scrambling condition. The conditions of the theorem are easily checked by locating the prime cyclic tapes with the algorithm given in Section 3.3. This result has some important implications in the design of o-stable probabilistic computers that have a cyclic behavior. It points out how one can design a nontrivial cyclic probabilistic computer that is zero-stable. Section 3.6 contains some zero-stability results in terms of the algebraic structure of NQD automata. Theorem 3.6.4 proves that any zero-reset automaton is zero-stable. Essentially, a zero-reset automaton consists of a finite number of quasi-definite subautomata. Each subautomaton can compute independently, and each can communicate with any other by means of the permutation structure. This structure gives a powerful computing ability to zero-reset automaton. The development in Section 3.6 proved that a monoid set $\mathcal{S}$ of $(n \times n)$ stochastic matrices with identity element U is a group, if each element $P \in \mathcal{S}$ has a corresponding reset element $P^r \in \mathcal{S}$ such that $P P^r \stackrel{o}{\sim} U$. This result is important in deciding whether or not a given set of stochastic matrices is a group. Since group automata are subsumed by zero-reset automata, we also know that group automata are zero-stable.

Chapter 4 gives several tests to decide the isolated cut-point problems. Theorem 4.1.1 proves for quasi-definite automata that the isolated cut-point problems can be decided with a bounded experiment. Theorem 4.1.2 gives sufficient conditions to imply that a quasi-definite automaton has no isolated cut points. Section 4.2 gives a neat view of tape response of an arbitrary probabilistic automaton. A response model is introduced which defines the entire response of a probabilistic automaton in terms of the response of the automaton to short tapes. A pseudo-closure operator is defined in terms of this response model which encloses the cut points which are not isolated. Conjecture 2.4.2 then indicates that the isolated cut-point problems can be decided in terms of this closure operator with finite experiments.

Let us conclude by pointing out some interesting and still open problems which merit further investigation. The bound given on the lengths of a prime cycling tape was obtained in Lemma 3.2.2 without placing any restrictions on the size of the alphabet. It would be important to see if one could lower the given bound by considering the constraints imposed by the alphabet size. The crucial idea in Theorem 3.5.1 is to prevent the minimal nonzero entry from having a zero limit prior to attaining the scrambling condition. It would be important to investigate extensions of this idea to a larger class of cyclic automata. The algebraic structure of NQD automata developed in Section 3.6 has some interesting implications on the computing behavior of NQD automata. It seems very attractive to pursue this approach to more general algebraic systems. The development on the isolated cut-point problem indicates, that it is reasonable to investigate certain properties of the response model that would decide the isolated cut-point problems. Finally, we express the need for a procedure for designing a probabilistic automaton to accept a given set of tapes, $T(\mathcal{P}, \lambda)$, with respect to the cut point λ.

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